Technical Record© #3 (in System and Control)

when I was with

Professor Katsuhisa Furuta

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by
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“Gandalf, Gandalf! Good gracious me! Not the wandering wizard that gave Old Took a pair of magic diamond studs that fastened themselves and never came undone till ordered? Not the felow who used to tell such wonderful tales at parties, about dragons and goblins and giants and the rescue of pricesses and the unexpected luck of widows’ sons? . . . Bless me, life used to be quite inter—I mean, you used to upset things badly in these parts once upon a time. I beg your pardon, but I had no idea you were still in business.”

J. R. R. Tolkien, *The Hobbit.*

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Introduction

This was supposed to be the Edition 1 of the #3 report presented to Professor Katsuhisa Furuta by Kittisak Tiyapan. Then I quitted my Ph.D. degree study. So the technical report became a technical record. The number still remained the same. Upto this moment there had already been three technical reports, namely Technical Report #1, #2, and #6. From now on I will call the rest by the name **Technical Record**. This is also good in a way that it will not be mistaken for a typical technical report published by an instution. I have no intention to claim this as a published item when it is not.

All simulation for this report was done by MATLAB Version 5.1.0.421 on a Solaris 2 machine named Hayate.

Tunnel diode circuit

Consider a Tunnel Diode circuit of Figure 1 [Kha96]

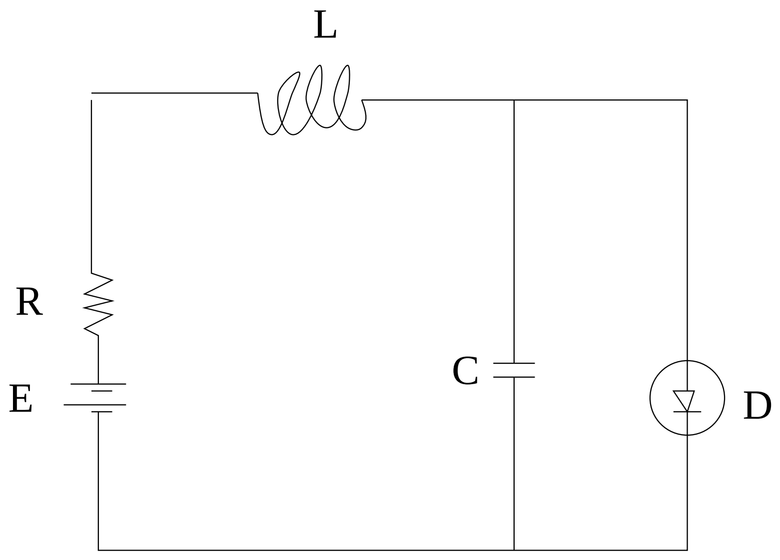


Figure 1: *Tunnel Diode Circuit.*

which in state space model is represented by

$$\begin{aligned} \dot{x_1} &= \frac{1}{C}\left[-h(x_1) + x_2\right] && (1) \\ \dot{x_2} &= \frac{1}{L}\left[-x_1 - Rx_2 + u\right]. && (2) \end{aligned}$$

Equation 1 and 2 are based on the assumption that both the capacitor C and the inductor L are linear and time-invariant. Their models are

$$i_C = C\frac{dv_C}{dt}, \quad v_L = L\frac{di_L}{dt}.$$

The tunnel diode is characterized by

$$i_R = h\left(v_R\right).$$

Then by Kirchhoff's current law,

$$i_C + i_R - i_L = 0 \quad \Rightarrow \quad i_C = -h\left(x_1\right) + x_2.$$

And by Kirchhoff's voltage law,

$$v_C - E + Ri_L + v_L = 0 \quad \Rightarrow \quad v_L = -x_1 - Rx_2 + u.$$

And where

$$x_1 = v_C, \quad x_2 = i_L, \quad u = E.$$

The equilibrium point correspond to the roots of

$$h\left(x_1\right) = \frac{E}{R} - \frac{1}{R}x_1.$$

Let the are $u = 3\ V$, $R = 5\ k\Omega$, $C = 6\ pF$ and $L = 7\ \mu H$. Then the state-space model with time measured in nanoseconds is (the current x_2 and $h(x_1)$ are in mA)

$$\begin{aligned} \dot{x_1} &= \frac{1}{6}\left[-h(x_1) + x_2\right] && (3) \\ \dot{x_2} &= \frac{1}{7}\left[-x_1 - 5x_2 + 3\right]. && (4) \end{aligned}$$

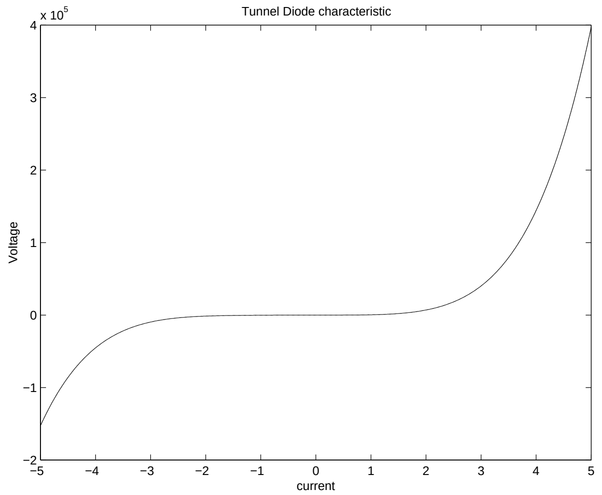
Let the characteristic of this diode be represented by

$$h(x_1) = 20x_1 - 100x_1^2 + 200x_1^3 + 200x_1^4 + 80x_1^5 \tag{5}$$

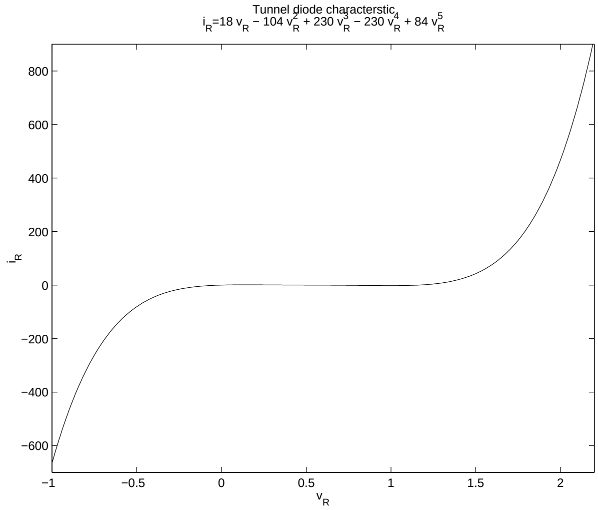
which is graphically shown as Figure 2 (a). Then let the characteristic be

$$h(x_1) = 18x_1 - 104x_1^2 + 230x_1^3 - 230x_1^4 + 84x_1^5 \tag{6}$$

which is Figure 2 (b). Figure 2 shows that the characteristic of the Tunnel Diode for Equation 5 and Equation 6 look similar graphically.



(a)



(b)

Figure 2: *Tunnel Diode characteristic*

Some knowledge about this system

To find equilibrium points obtain Equation 7 and 8 from Equation 3 and 4 respectively.

$$0 = \frac{1}{6}\left[-h(x_1) + x_2\right] \tag{7}$$

$$0 = \frac{1}{7}\left[-x_1 - 5x_2 + 3\right]. \tag{8}$$

Solving Equation 7, 8, and 5 leads to the equilibrium points

$$\left({}^{eq}x_1, \frac{-3 - {}^{eq}x_1}{5}\right), \tag{9}$$

where ${}^{eq}x_1$ is the x_1 at a equilibrium point and can be obtained by solving the equation

$$-3 + 101\,x_1 - 500\,x_1^2 + 1000\,x_1^3 + 1000\,x_1^4 + 400\,x_1^5 = 0. \tag{10}$$

or the equation

$$18x_1 - 104x_1^2 + 230x_1^3 - 230x_1^4 + 84x_1^5 - \frac{3}{5} + \frac{1}{5}x_1 = 0 \tag{11}$$

for the case when the characteristic of the diode is Equation 5 and 6 respectively. The solution to Equation 10 is

$$x_1 = -1.45 \pm 1.22\,i, \quad 0.0355, \quad 0.184 \pm 0.157\,i,$$

while that for Equation 11 is

$$x_1 \approx 0.042257, \quad 0.314562, \quad 0.603715 \pm 0.305466\,i, \quad 1.173846.$$

In the first case the only equilibrium point is the point where x_1 is a real root, ie $(0.0355, -0.6071)$, while in the second case there are three equilibrium points, ie $(0.042257, -0.6085)$, $(0.314562, -0.6629)$, and $(1.1738, -0.8348)$.

From Equation 3 to 5 the state equation of the system can be written as

$$\begin{array}{rcccl} \dot{x}_1 & = & f_1(x) & = & \frac{1}{6}\left(-20x_1 + 100x_1^2 - 200x_1^3 - 200x_1^4 - 80x_1^5 + x_2\right) \\ \dot{x}_2 & = & f_2(x) & = & \frac{1}{7}\left[-x_1 - 5x_2 + 3\right] \end{array}. \tag{12}$$

Find the Jacobian matrix from

$$\frac{\partial f(x)}{\partial x} = \begin{bmatrix} \frac{\partial f_1(x)}{\partial x_1} & \frac{\partial f_1(x)}{\partial x_2} \\ \frac{\partial f_2(x)}{\partial x_1} & \frac{\partial f_2(x)}{\partial x_2} \end{bmatrix} \tag{13}$$

and

$$\frac{\partial f_1(x)}{\partial x_1} = \frac{-20x_1 + 100x_1^2 - 200x_1^3 - 200x_1^4 - 80x_1^5 + x_2}{6}, \frac{\partial f_1(x)}{\partial x_2} = \frac{1}{6}, \frac{\partial f_2(x)}{\partial x_1} = -\frac{1}{7}, \frac{\partial f_1(x)}{\partial x_2} = -\frac{5}{7},$$

which leads to

$$A = \left.\frac{\partial f(x)}{\partial x}\right|_{eq} = \begin{bmatrix} -2.2821 & 0.1667 \\ -0.1429 & -0.7143 \end{bmatrix}.$$

From which the eigenvalues can be obtained to be at -2.2668 and -0.7296. Both are negative, therefore are stable nodes.

Figure 3 shows the phase plane of the diode of Equation 3 to 5. The initial conditions used in simulating Figure 3 was $\{\{-5,5\}\times\{-5\leq\mathcal{I}\leq 5\}\}\cup\{\{-5\leq\mathcal{I}\leq 5\}\times\{-5,5\}\}$.

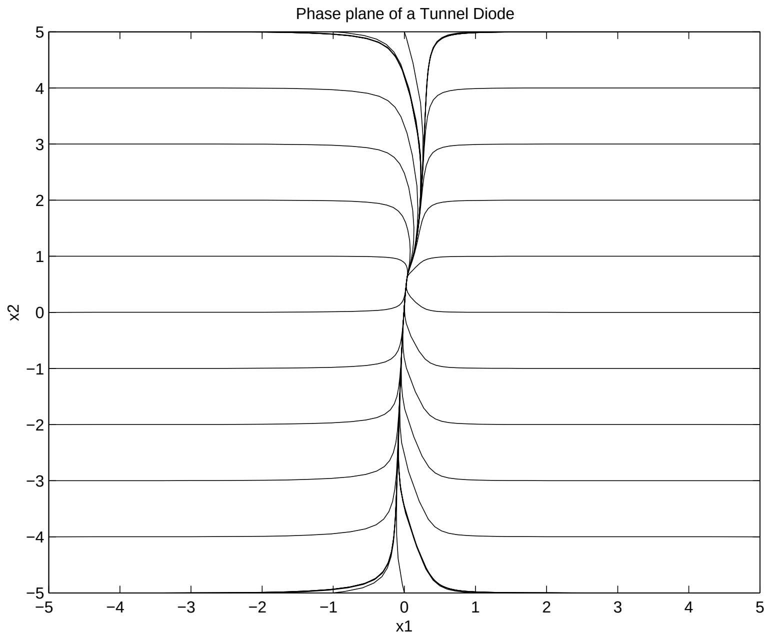


Figure 3: *Phase plane, Tunnel Diode.*

With a sign function in feedback

Figure 4 shows the phase plane of the diode when $u = \frac{3}{7} + \mathrm{sgn}(\mathrm{x}_1 + \mathrm{x}_2)$.

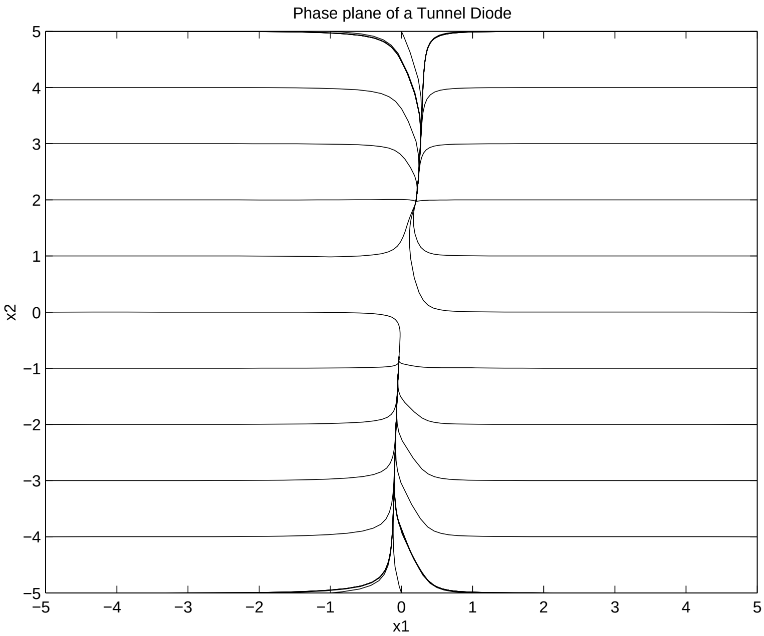


Figure 4: *Phase plane with sign state feed back, Tunnel Diode.*

With a sign function and state variables in the feedback

Figure 4 shows the phase plane of the diode when $u = \frac{3}{7} + \mathrm{sgn}(\mathrm{x}_1 + \mathrm{x}_2) - (\mathrm{x}_1 + \mathrm{x}_2)$.

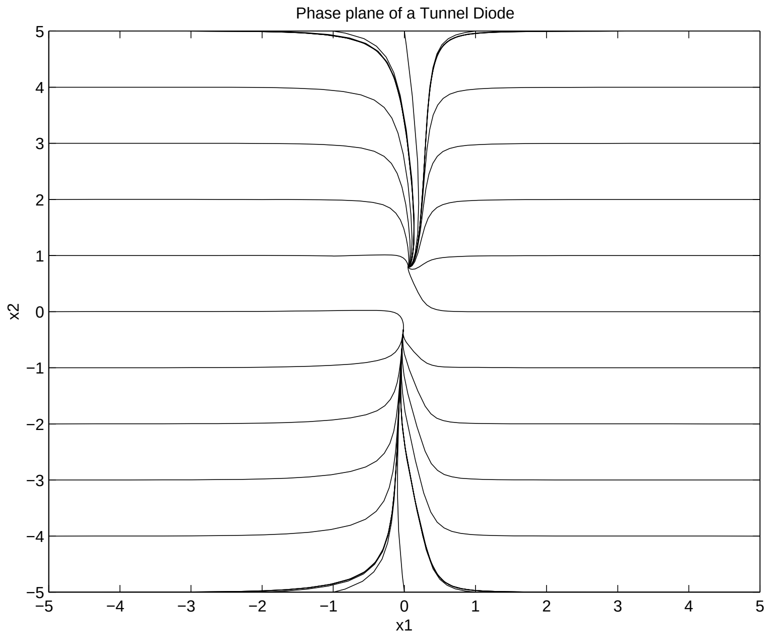


Figure 5: *Phase plane with sign and state feed back, Tunnel Diode.*

Backstepping control design

Consider the case where the characteristic of the diode is not known, then $h(x_1)$ is uncertain, but is in the form of a fifth order polynomial passing through the origin (ie in the form of Equation 5 and 6 with coefficients uncertain but with bounds.) Then Equation 3 and 4 do not satisfy the matching condition since there is an uncertain term in Equation 3 where there is no control input. The resistance R can vary. The capacitance C and the inductance L are assumed to be exactly known. In other words, the system can be written as

$$\dot{x}_1 = \theta_1 x_1 + \theta_2 x_1^2 + \theta_3 x_1^3 + \theta_4 x_1^4 + \theta_5 x_1^5 + s x_2 \quad (14)$$

$$\dot{x}_2 = -r x_1 + \theta_6 x_2 + r u, \quad (15)$$

where

$$|\theta_1| < a, |\theta_2| < b, |\theta_3| < c, |\theta_4| < d, |\theta_5| < e, \theta_6 < p$$

and

$$a, b, c, d, e, p, r, s, \theta_6 \in \mathcal{R}^+.$$

Notice that when the characteristic of the tunnel diode is described by

$$h(x_1) = \kappa_1 x_1 + \kappa_2 x_1^2 + \kappa_3 x_1^3 + \kappa_4 x_1^4 + \kappa_5 x_1^5,$$

then

$$\theta_1 = -\frac{\kappa_1}{C},\ \theta_2 = -\frac{\kappa_2}{C},\ \theta_3 = -\frac{\kappa_3}{C},\ \theta_4 = -\frac{\kappa_4}{C},\ \theta_5 = -\frac{\kappa_5}{C},\ s = \frac{1}{C},\ \theta_6 = -\frac{R}{L},\ r = \frac{1}{L}.$$

Let the parameters be as described in page 2 and $h(x_1)$ as in Equation 5, the result is shown in Figure 6. Parameters used for simulating Figure 6 may be summarized to comply with Equation

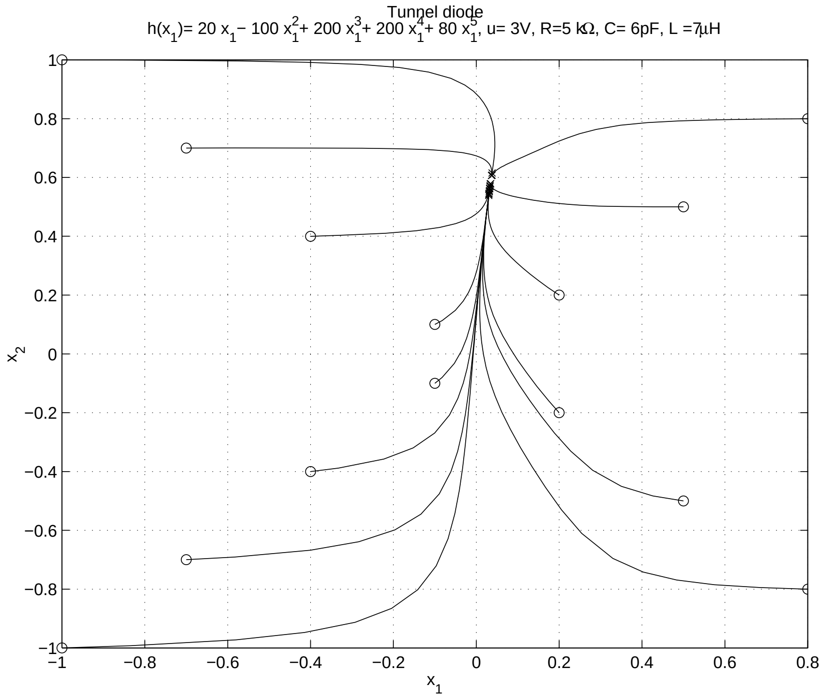


Figure 6: *Tunnel Diode,* $h(x_1) = 20x_1 - 100x_1^2 + 200x_1^3 + 200x_1^4 + 80x_1^5$*,* $u = 3V$*,* $R = 5k\Omega$*,* $C = 6pF$*,* $L = 7\mu H$

14, 15 and as

$$\kappa_1 =, 20\ \kappa_2 = -100,\ \kappa_3 = 200,\ \kappa_4 = 200,\ \kappa_5 = 80,\ E = 3V,\ R = 5\,k\Omega,\ C = 6\,pF,\ L = 7\,\mu H,$$

time is measured in nanoseconds and current in mA. The time of simulation for Figure 6 is 5 seconds. The initial conditions are Initial Condition Set 1.

Initial Condition Set 1 *The set of points* $(x_1(0), x_2(0))$ *created by*

```
n = 0;
for i = -1 to 1 steps 0.3
  n = n + 1, x1(0) = i and x2(0) = i are the n_th initial conditions
  n = n + 1, x1(0) = i and x2(0) = -i are the n_th initial conditions
endfor
```

Then let the parameters be the same but let $h(x_1)$ be as in Equation 6 instead, the result is shown in Figure 7. Parameters used for simulating Figure 7 may be summarized to comply with Equation 14, 15 and as

$$\kappa_1 =, 18\ \kappa_2 = -104,\ \kappa_3 = 230,\ \kappa_4 = -230,\ \kappa_5 = 84,\ E = 3V,\ R = 5\,k\Omega,\ C = 6\,pF,\ L = 7\,\mu H.$$

The initial conditions were Initial Condition Set 1. The time of simulation is 5 seconds for half of the initial conditions set (the half with (i, i) according to Initial Condition Set 1) and 10 seconds for the rest (ie the half with $(i, -i)$.)

Suppose that the uncertainties within this system are described as follows,

$$\begin{array}{lllllllllll}
-5 & \leq & \kappa_1 & \leq & 30 & , & -200 & \leq & \kappa_2 & \leq & 0 \\
10 & \leq & \kappa_3 & \leq & 500 & , & -500 & \leq & \kappa_4 & \leq & 500 \\
-10 & \leq & \kappa_5 & \leq & 100 & , & 1\,k\Omega & \leq & R & \leq & 10\,k\Omega
\end{array}$$

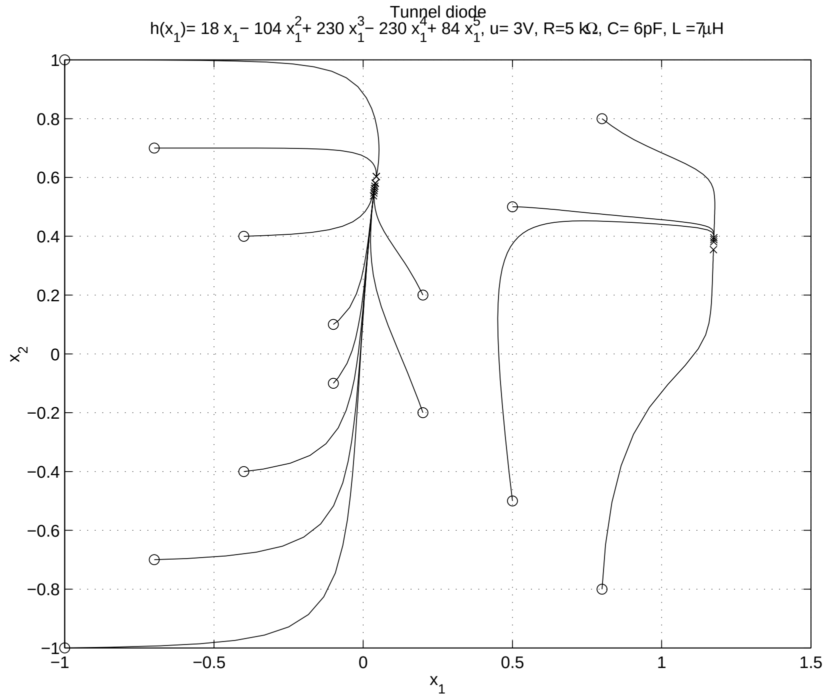


Figure 7: *Tunnel Diode,* $h(x_1) = 18x_1 - 104x_1^2 + 230x_1^3 - 230x_1^4 + 84x_1^5$, $u = 3V$, $R = 5k\Omega$, $C = 6pF$, $L = 7\mu H$

which could be rewritten (though with somewhat more stringent a requirement) as

$$\begin{array}{rclcrclcrcl} |\theta_1| & \leq & 5 & , & |\theta_2| & \leq & 35 & , & |\theta_3| & \leq & 85 \\ |\theta_4| & \leq & 85 & , & |\theta_5| & \leq & 20 & , & |\theta_6| & \leq & \frac{10}{7} = 1.4286 \end{array}$$

The following theorem is actually Lemma 13.3 in Khalil [Kha96].

Theorem 1 *Consider the system*

$$\dot{\eta} = f(\eta) + g(\eta)\xi + \delta_\eta(\eta, \xi) \tag{16}$$
$$\dot{\xi} = f_a(\eta, \xi) + g_a(\eta, \xi)u + \delta_\xi(\eta, \xi), \tag{17}$$

defined on a domain $\mathcal{D} \subset \mathcal{R}^{n+1}$ *that contains the origin. Let the uncertainty satisfies* $\forall(\eta, \xi) \in \mathcal{D}$,

$$\|\delta_\eta(\eta, \xi)\|_2 \leq \alpha_1 \|\eta\|_2 \tag{18}$$
$$|\delta_\xi(\eta, \xi)| \leq \alpha_2 \|\eta\|_2 + \alpha_3|\xi| \tag{19}$$

Let $\phi(\eta)$ *be a stabilizing state feedback control for Equation 16 that satisfies*

$$V_a(\eta, \xi) = V(\eta) + \frac{1}{2}[\xi - \phi(\eta)]^2 \tag{20}$$

and $V(\eta)$ *be a Lyapunov function that satisfies* $\forall \eta \in \mathcal{D}$

$$|\phi(\eta)| \leq \alpha_4\|\eta\|_2, \qquad \left\|\frac{\partial\phi}{\partial\eta}\right\|_2 \leq \alpha_5 \tag{21}$$

Then the state feedback control

$$u = \frac{1}{g_a}\left[\frac{\partial\phi}{\partial\eta}(f + g\xi) - \frac{\partial V}{\partial\eta}g - f_a - k(\xi - \phi)\right], \quad k > 0, \tag{22}$$

with k sufficiently large, stabilizes the origin of Equation 16 and 17. Moreover, if all the assumptions hold globally and $V(\eta)$ is radially unbounded, the origin will be globally asymptotically stable. ♠

(see Khalil [Kha96] Lemma 13.3 for the proof)

From Theorem 1 Equation 14 and 15 is of the form of Equation 16 and 17 respectively, where

$$\begin{array}{lcl} \eta & = & x_1 \\ f(\eta) & = & 0 \\ \delta_\eta & = & \theta_1 x_1 + \theta_2 x_1^2 + \theta_3 x_1^3 + \theta_4 x_1^4 + \theta_5 x_1^5 \\ g_a(\eta,\xi) & = & r \end{array} \quad \begin{array}{lcl} ,\ \xi & = & x_2 \\ ,\ g(\eta) & = & s \\ ,\ f_a(\eta,\xi) & = & r x_1 \\ ,\ \delta_\xi & = & \theta_6 x_2 \end{array}$$

The uncertainty terms are

$$\delta_1 = \delta_\eta = \theta_1 x_1 + \theta_2 x_1^2 + \theta_3 x_1^3 + \theta_4 x_1^4 + \theta_5 x_1^5, \quad \text{and} \quad \delta_2 = \delta_\xi = \theta_6 x_2.$$

For $|x_1| \le \varsigma$ and $|x_2| \le \sigma$

$$\begin{aligned} |\delta_1| &\le 5\varsigma + 35\varsigma^2 + 85\varsigma^3 + 85\varsigma^4 + 20\varsigma^5 \\ &\le \left(5 + 35\varsigma + 85\varsigma^2 + 85\varsigma^3 + 20\varsigma^4\right)|x_1| \end{aligned}$$

and
$$|\delta_2| \le 1.4286\sigma \le 1.4286\,|x_2|\,.$$

Consider Equation 14 as a system and view x_2 as a control input (notice that $s = \frac{1}{6}$, $r = \frac{1}{7}$,)

$$x_2 = \phi_1(x_1) = -k_1 x_1$$

and
$$V_1(x_1) = \frac{1}{2}x_1^2.$$

Then
$$\begin{aligned} \dot{V}_1 &= s x_1 \phi(x_1) + \theta_1 x_1^2 + \theta_2 x_1^3 + \theta_3 x_1^4 + \theta_4 x_1^5 + \theta_5 x_1^6 \\ &\le -\left(s k_1 - \left(5 + 35\varsigma + 85\varsigma^2 + 85\varsigma^3 + 20\varsigma^4\right)\right) x_1^2. \end{aligned}$$

Choose
$$k_1 = \frac{1 + \theta_1 + \theta_2\varsigma + \theta_3\varsigma^2 + \theta_4\varsigma^3 + \theta_5\varsigma^4}{s} = \frac{1 + 5 + 35\varsigma + 85\varsigma^2 + 85\varsigma^3 + 20\varsigma^4}{s}.$$

Then from Equation 20 and 22

$$\begin{aligned} u &= 7\left[-k_1\frac{x_2}{6} - \left(\frac{1}{6} + \frac{1}{7}\right)x_1 - k\left(x_2 + k_1 x_1\right)\right] \\ &= -1.1669 k_1 x_2 - 2.1665 x_1 - 0.1429 k\left(x_2 + k_1 x_1\right), \qquad \text{and} \\ V_a &= \frac{1}{2}\left[x_1^2 + \left(x_2 + k_1 x_1\right)^2\right] \end{aligned}$$

The derivative of the Lyapunov function is

$$\dot{V}_a = \frac{1}{2}\left[2x_1\dot{x}_1 + 2\left(k_1 x_1 + x_2\right)\left(k_1\dot{x}_1 + \dot{x}_2\right)\right].$$

Restrict the analysis to the set
$$\Omega_c = \left\{x \in \mathcal{R}^2 \mid V_a(x) \le c\right\}.$$

Notice that
$$c = \frac{1}{2}\left[\varsigma^2 + \left(\sigma + \frac{(1 + \max|\theta_1| + \max|\theta_2|\varsigma + \max|\theta_3|\varsigma^2 + \max|\theta_4|\varsigma^3 + \max|\theta_5|\varsigma^4)}{s}\varsigma\right)^2\right].$$

Substituting

$$\begin{aligned}
\dot{x}_1 &= \theta_1 x_1 + \theta_2 x_1^2 + \theta_3 x_1^3 + \theta_4 x_1^4 + \theta_5 x_1^5 - k_1 s x_1 \\
\dot{x}_2 &= -2rx_1 - sx_1 - k_1 s x_2 + \theta_6 x_2 - k\left(k_1 x_1 + x_2\right)
\end{aligned}$$

gives

$$\begin{aligned}
\dot{V}_a &= \frac{1}{2}\Big[2x_1\left(-sk_1x_1 + \theta_1 x_1 + \theta_2 x_1^2 + \theta_3 x_1^3 + \theta_4 x_1^4 + \theta_5 x_1^5\right) \\
&\quad +2\left(k_1x_1 + x_2\right)\left(-2rx_1 - sx_1 + k_1\left(-sk_1x_1 + \theta_1 x_1 + \theta_2 x_1^2 + \theta_3 x_1^3 + \theta_4 x_1^4 + \theta_5 x_1^5\right)\right. \\
&\quad \left.-sk_1x_2 + \theta_6 x_2 - k\left(k_1x_1 + x_2\right)\right)\Big] \\
&= -x_1^2 + \left(k_1x_1 + x_2\right)\left(-2r - s - k_1^2 s\right)x_1 + \left(k_1x_1 + x_2\right)\left(\theta_1 + \theta_2 x_1 + \theta_3 x_1^2 + \theta_4 x_1^3 + \theta_5 x_1^4\right)x_1^2 k_1 \\
&\quad + \left(k_1x_1 + x_2\right)\left(-sk_1 + \theta_6\right)x_2 - k\left(k_1x_1 + x_2\right)^2 \\
&\leq \left(k_1x_1 + x_2\right)\left(-2r - s - k_1^2 s\right)x_1 + \left|k_1x_1 + x_2\right|\left(\theta_1 + \theta_2\varsigma + \theta_3\varsigma^2 + \theta_4\varsigma^3 + \theta_5\varsigma^4\right)\varsigma\left(k_1x_1 + x_2 - x_2\right) \\
&\quad + \left(k_1x_1 + x_2\right)\left(-sk_1 + \theta_6\right)x_2 - k\left(k_1x_1 + x_2\right)^2 \\
&\leq \left(k_1x_1 + x_2\right)\left(-2r - s - k_1^2 s\right)x_1 + \left|k_1x_1 + x_2\right|\left(-1 - \left(1 + \varsigma\right)\left(\theta_1 + \theta_2\varsigma + \theta_3\varsigma^2 + \theta_4\varsigma^3 + \theta_5\varsigma^4\right)\varsigma + \theta_6\right) \\
&\quad - \left(k - \left(\theta_1 + \theta_2\varsigma + \theta_3\varsigma^2 + \theta_4\varsigma^3 + \theta_5\varsigma^4\right)\varsigma\right)\left(k_1x_1 + x_2\right)^2
\end{aligned}$$

Choose

$$\begin{aligned}
k &> \left(\max|\theta_1| + \max|\theta_2|\varsigma + \max|\theta_3|\varsigma^2 + \max|\theta_4|\varsigma^3 + \max|\theta_5|\varsigma^4\right)\varsigma + \left(-2r - s - k_1^2 s\right)^2 \\
&\quad + \left[\max|\theta_6| - 1 - \left(1 + \varsigma\right)\left(\max|\theta_1| + \max|\theta_2|\varsigma + \max|\theta_3|\varsigma^2 + \max|\theta_4|\varsigma^3 + \max|\theta_5|\varsigma^4\right)\right]^2
\end{aligned}$$

would ensure that the origin is exponentially stable (see Khalil [Kha96] Corollary 3.4), with Ω_c contained in the region of attraction.

Then with the values of parameter above and

$$\varsigma = 0.1, \quad \sigma = 0.2$$

the gain k becomes

$$k_1 = 44.62 \Rightarrow k = 1.105 \times 10^5.$$

To show graphically the result, consider the system where

$$\kappa_1 = 22,\ \kappa_2 = -180,\ \kappa_3 = 440,\ \kappa_{-490} =,\ \kappa_5 = 90,\ E = 1\,\mathrm{V},\ R = 9\,\mathrm{k\Omega},\ C = 6\,\mathrm{pF},\ L = 7\,\mu H,\ \varsigma = 0.1,\ \sigma$$

The initial conditions used are Initial Condition Set 2 and the simulation time is 5 seconds.

Initial Condition Set 2 *The set of points* $(x_1(0), x_2(0))$ *created by*

```
n = 0;
for i = -0.5 to 0.5 steps 0.11
  n = n + 1, x1(0) = i and x2(0) = i are the n_th initial conditions
  n = n + 1, x1(0) = i and x2(0) = -i are the n_th initial conditions
endfor
```

The result is shown in Figure 8 Apply a control input as shown in Equation 23, k_1, k as has been calculated above. The simulation result is shown in Figure 9 (Initial Condition Set 2 and simulation time 5 seconds.)

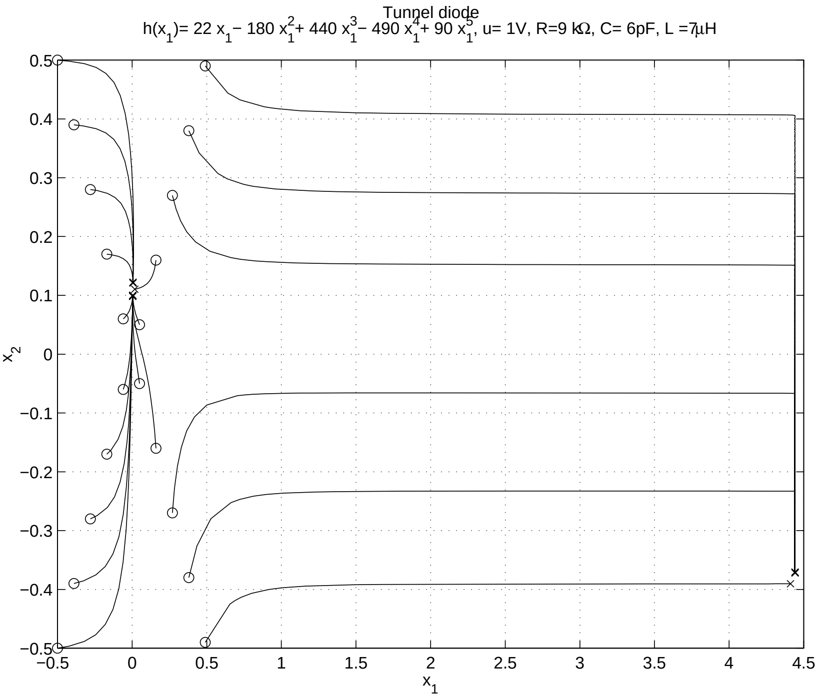


Figure 8: *Tunnel Diode,* $h(x_1) = 22x_1 - 180x_1^2 + 440x_1^3 - 490x_1^4 + 90x_1^5$, $u = 1V$, $R = 9k\Omega$, $C = 6pF$, $L = 7\mu H$

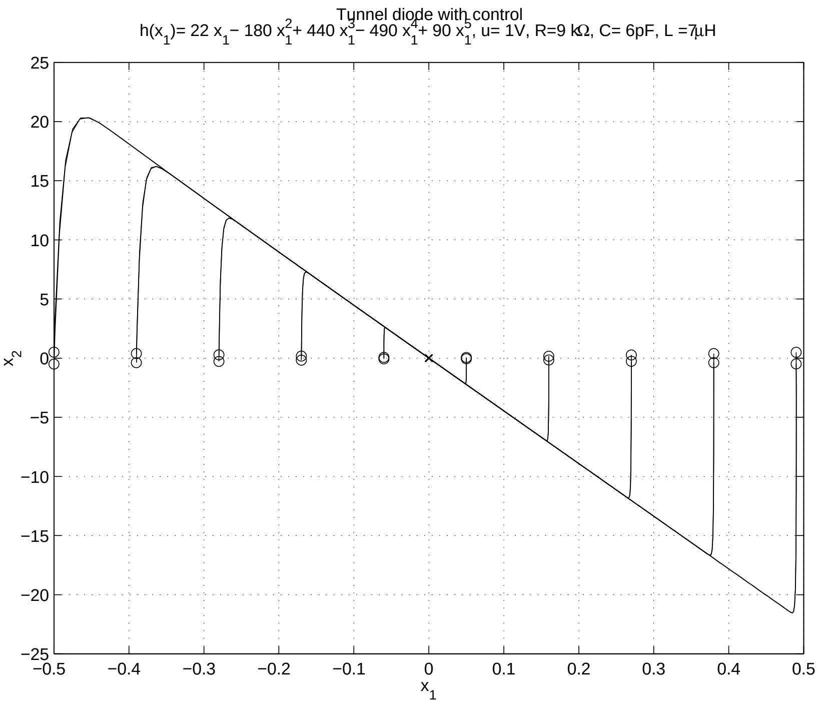


Figure 9: *Tunnel Diode with control,* $h(x_1) = 22x_1 - 180x_1^2 + 440x_1^3 - 490x_1^4 + 90x_1^5$, $u = 1V$, $R = 9k\Omega$, $C = 6pF$, $L = 7\mu H$

Lorenz attractor

13^{th} **May 1998**
Consider a Lorenz attractor

$$\begin{aligned} \dot{x_1} &= \sigma\left(x_2 - x_1\right) && (23) \\ \dot{x_2} &= x_1\left(1+\lambda - x_3\right) - x_2 && (24) \\ \dot{x_3} &= x_1 x_2 - b x_3. && (25) \end{aligned}$$

This system has got three equilibrium points, one of which is the origin and the other two are symmetrically positioned with respect to each other. To find the equilibrium points, let $\dot{x} = 0$ which leads to

$$\begin{aligned} 0 &= \sigma\left(x_2 - x_1\right) && (26) \\ 0 &= x_1\left(1+\lambda - x_3\right) - x_2 && (27) \\ 0 &= x_1 x_2 - b x_3, && (28) \end{aligned}$$

from which the equilibrium points are

$$(x_1, x_2, x_3) = (0,0,0), \quad \left(-\sqrt(b)\sqrt(\lambda), -\sqrt(b)\sqrt(\lambda), \lambda\right), \quad \left(\sqrt(b)\sqrt(\lambda), \sqrt(b)\sqrt(\lambda), \lambda\right))$$

The equilibrium point at the origin is unstable and the other two will be stable if the condition

$$\lambda\left(b+1-\sigma\right) + \left(\sigma+1\right)\left(\sigma+b+1\right) > 0 \tag{29}$$

is met.

Simulation result of the system

Results from simulations done on this system are shown from Figure 10 to Figure 12. Figure 10 is the system with

$$\sigma = 3, \quad \lambda = 5, \quad b = 1,$$

(Equation 29 is satisfied) and the initial conditions were

1. for (a), $(x_1, x_2, x_3) = (1,1,1)$,
2. for (b), $(x_1, x_2, x_3) \in \{-5 \le \mathcal{I}^+ \le 5\} \times \{1\} \times \{-5, 5\} \bigcup \{-5, 5\} \times \{1\} \times \{-5 \le \mathcal{I}^+ \le 5\}$,
3. for (c), $(x_1, x_2, x_3) \in \{-5 \le \mathcal{I}^+ \le 5\} \times \{-5, 5\} \times \{1\} \bigcup \{-5, 5\} \times \{-5 \le \mathcal{I}^+ \le 5\} \times \{1\}$,
4. for (d), $(x_1, x_2, x_3) \in \{1\} \times \{-5 \le \mathcal{I}^+ \le 5\} \times \{-5, 5\} \bigcup \{1\} \times \{-5, 5\} \times \{-5 \le \mathcal{I}^+ \le 5\}$, and
5. for (e), along each of the four diagonal lines.

The CPU time in the case of Figure 10(e) was 1.07 seconds, and its file size is 98.88 KB.

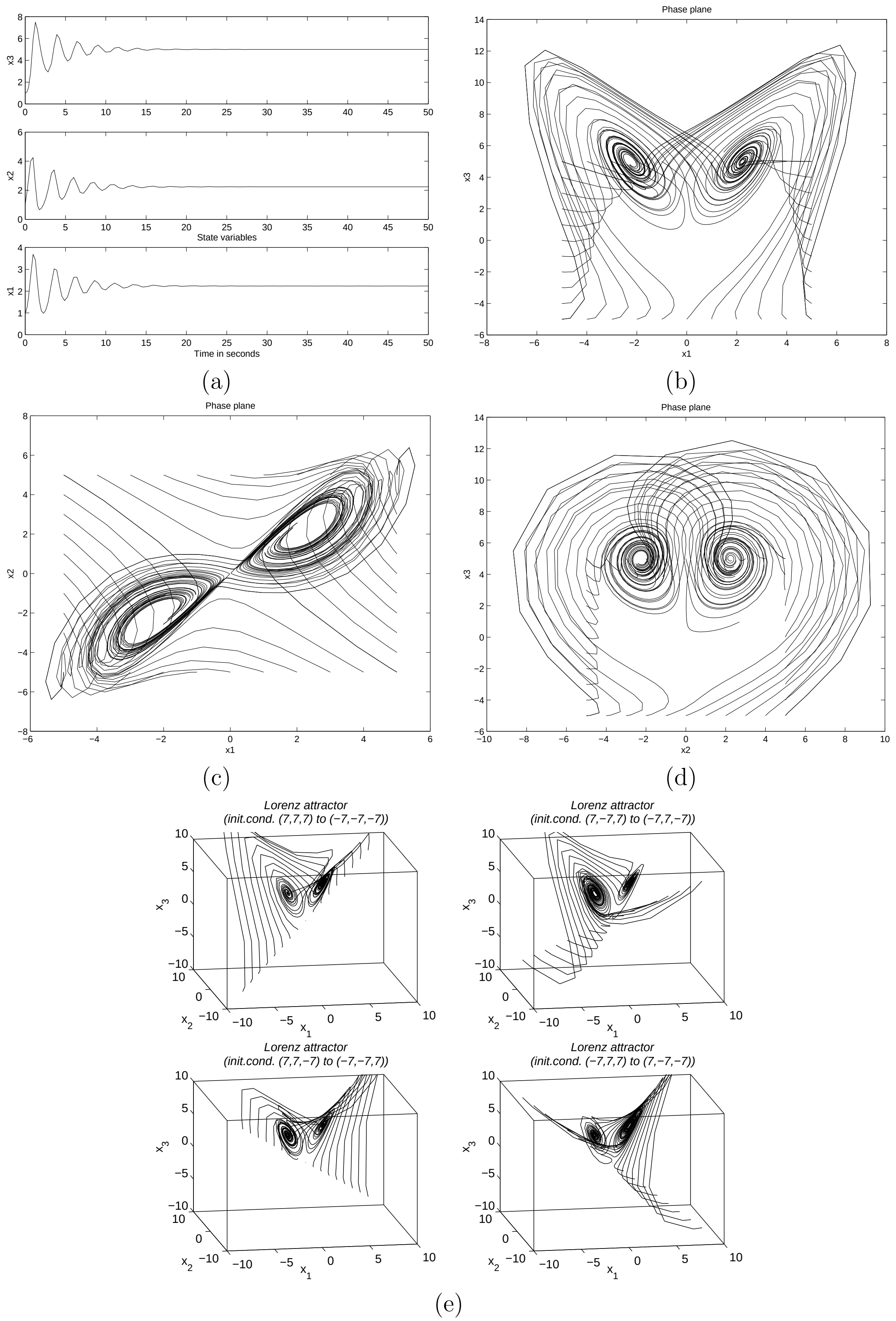


Figure 10: *Lorenz attractor ($\sigma = 3$, $\lambda = 5$, $b = 1$), (a) State variables, (b) State plane x_1 v.s. x_3, (c) State plane x_1 v.s. x_2, (d) State plane x_2 v.s. x_3, (e) State space*

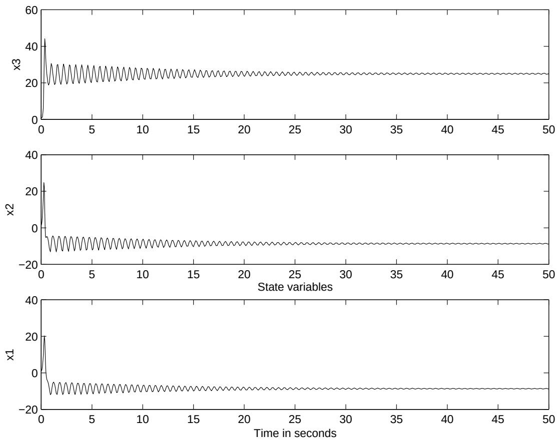
Figure 11 was from the system with

$$\sigma = 15, \quad \lambda = 25, \quad b = 3,$$

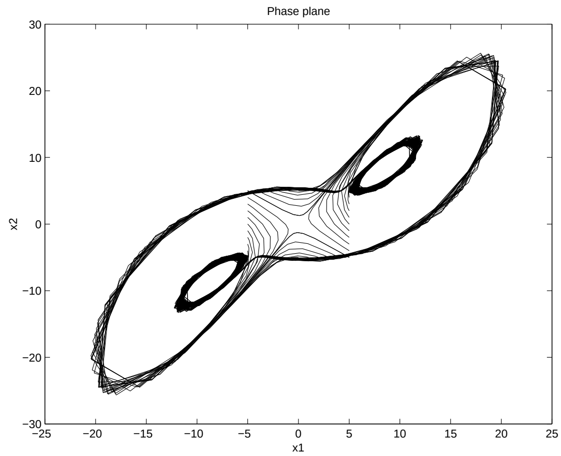
(Equation 29 is satisfied) with initial

1. for (a), $(x_1, x_2, x_3) = (1, 1, 1)$,
2. for (b), $(x_1, x_2, x_3) \in \{-5 \leq \mathcal{I}^+ \leq 5\} \times \{-5, 5\} \times \{1\} \bigcup \{-5, 5\} \times \{-5 \leq \mathcal{I}^+ \leq 5\} \times \{1\}$,
3. for (c), $(x_1, x_2, x_3) \in \{1\} \times \{-5 \leq \mathcal{I}^+ \leq 5\} \times \{-5, 5\} \bigcup \{1\} \times \{-5, 5\} \times \{-5 \leq \mathcal{I}^+ \leq 5\}$,
4. for (d), $(x_1, x_2, x_3) \in \{-5 \leq \mathcal{I}^+ \leq 5\} \times \{1\} \times \{-5, 5\} \bigcup \{-5, 5\} \times \{1\} \times \{-5 \leq \mathcal{I}^+ \leq 5\}$, and
5. for (e), along each of the four diagonal lines.

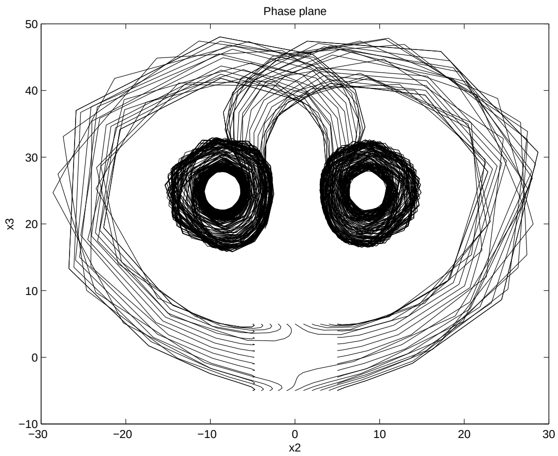
The CPU time used in simulating Figure 11 (e) was 2.32 seconds, and its file size is 217.24 KB.



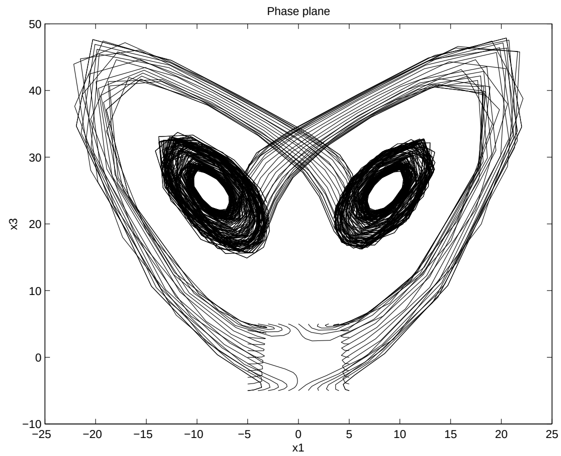
(a)



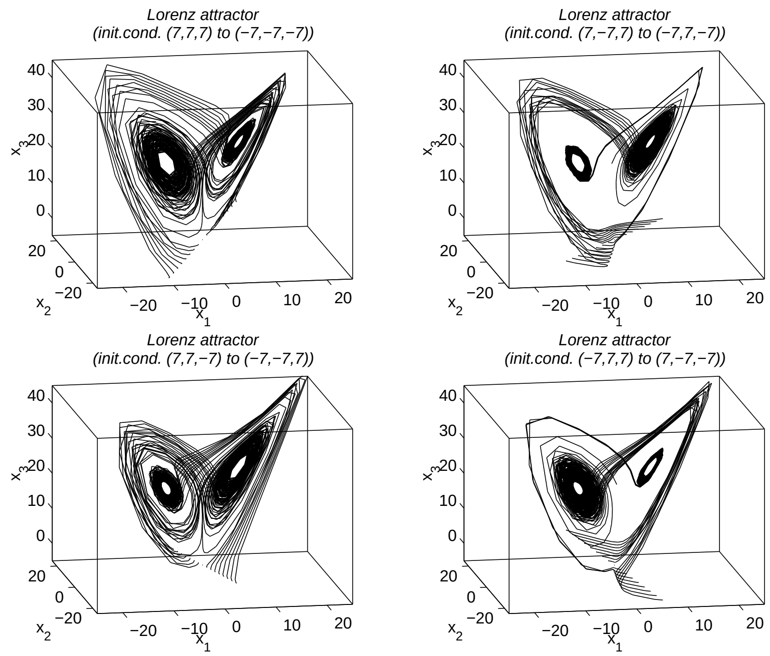
(b)



(c)



(d)



multicolumn2c(e)

Figure 11: *Lorenz attractor ($\sigma = 15$, $\lambda = 25$, $b = 3$), (a) State variables, (b) State plane x_1 v.s. x_2, (c) State plane x_2 v.s. x_3, (d) State plane x_1 v.s. x_3, (e) State space*

Figure 12 shows the result from a system having

$$\sigma = 9, \quad \lambda = 30, \quad b = 1,$$

(Equation 29 is not satisfied) together with the initial conditions as shown in page 15. The CPU time used during the simulation for Figure 12 (e) was 2.01 seconds, and its file size 196.51 KB.

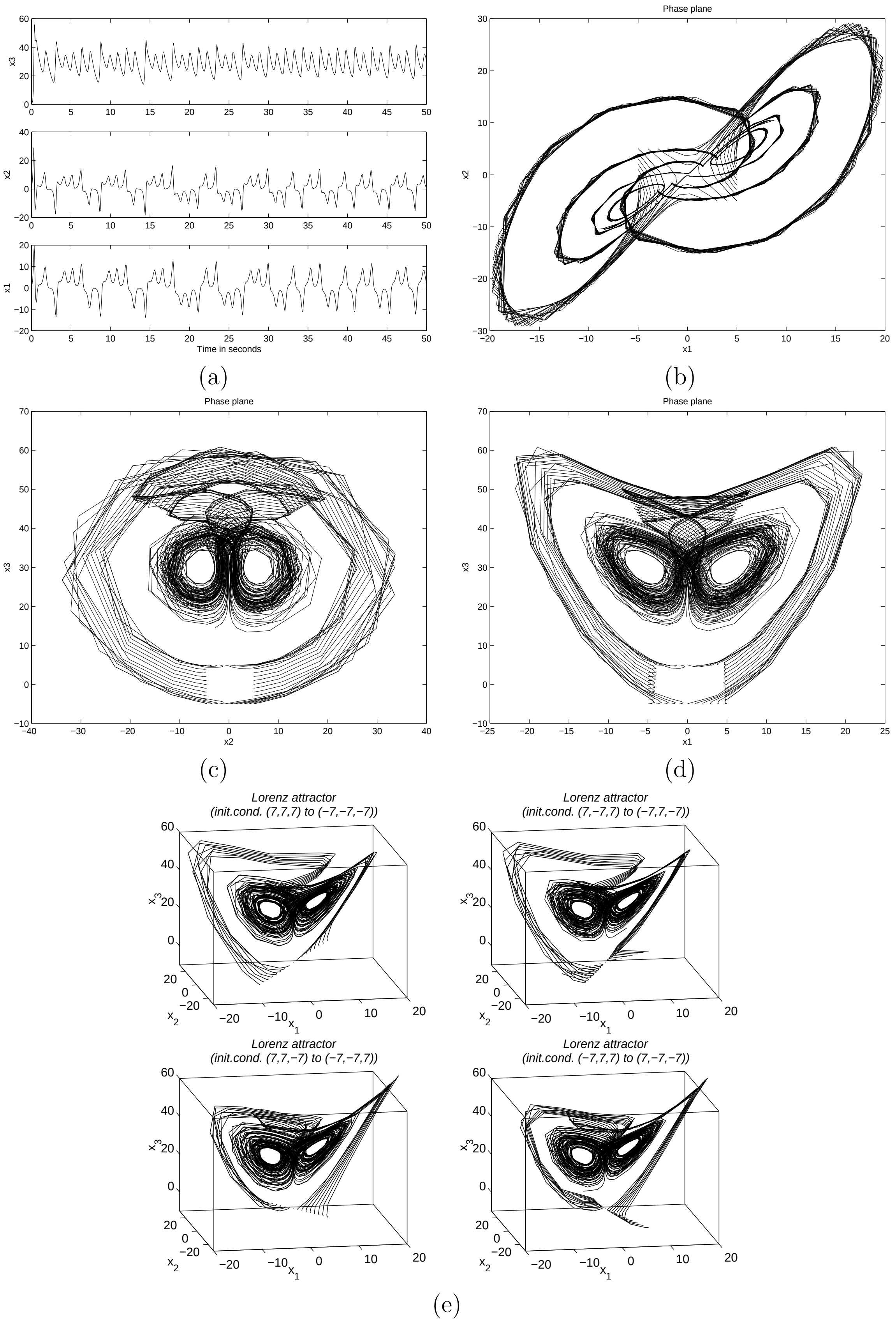


Figure 12: *Lorenz attractor ($\sigma = 9$, $\lambda = 30$, $b = 1$), (a) State variables (b)State plane x_1 v.s. x_2, (c) State plane x_2 v.s. x_3, (d) State plane x_1 v.s. x_3, (e) State space*

Lorenz attractor with control

Add an input into Equation 25 so that the system is now

$$\dot{x_1} = \sigma (x_2 - x_1) \tag{30}$$
$$\dot{x_2} = x_1 (1 + \lambda - x_3) - x_2 \tag{31}$$
$$\dot{x_3} = x_1 x_2 - b x_3 + u. \tag{32}$$

To demonstrate the effect of a sign function on the response let

$$u = K \operatorname{sgn} (\mathrm{x}_1 + \mathrm{x}_2 + \mathrm{x}_3). \tag{33}$$

(Errata: the initial condition labels written on Figure 10 should be the same as that of Figure 14, ie. all 7's should be replaced by 8's)

Let $K = -1$ and the input now becomes

$$u = -\operatorname{sgn} (\mathrm{x}_1 + \mathrm{x}_2 + \mathrm{x}_3). \tag{34}$$

Simulated result with this input is shown in Figure 13. Here the conditions were

$$\sigma = 3, \quad \lambda = 5, \quad b = 1,$$

(Equation 29 is satisfied) and the states at the start of simulation were the same as those initial conditions shown in page 15. Figure 13 (c) has got $t_{CPU} = 19.96$ seconds and the size of the file containing it is 1.87 MB.

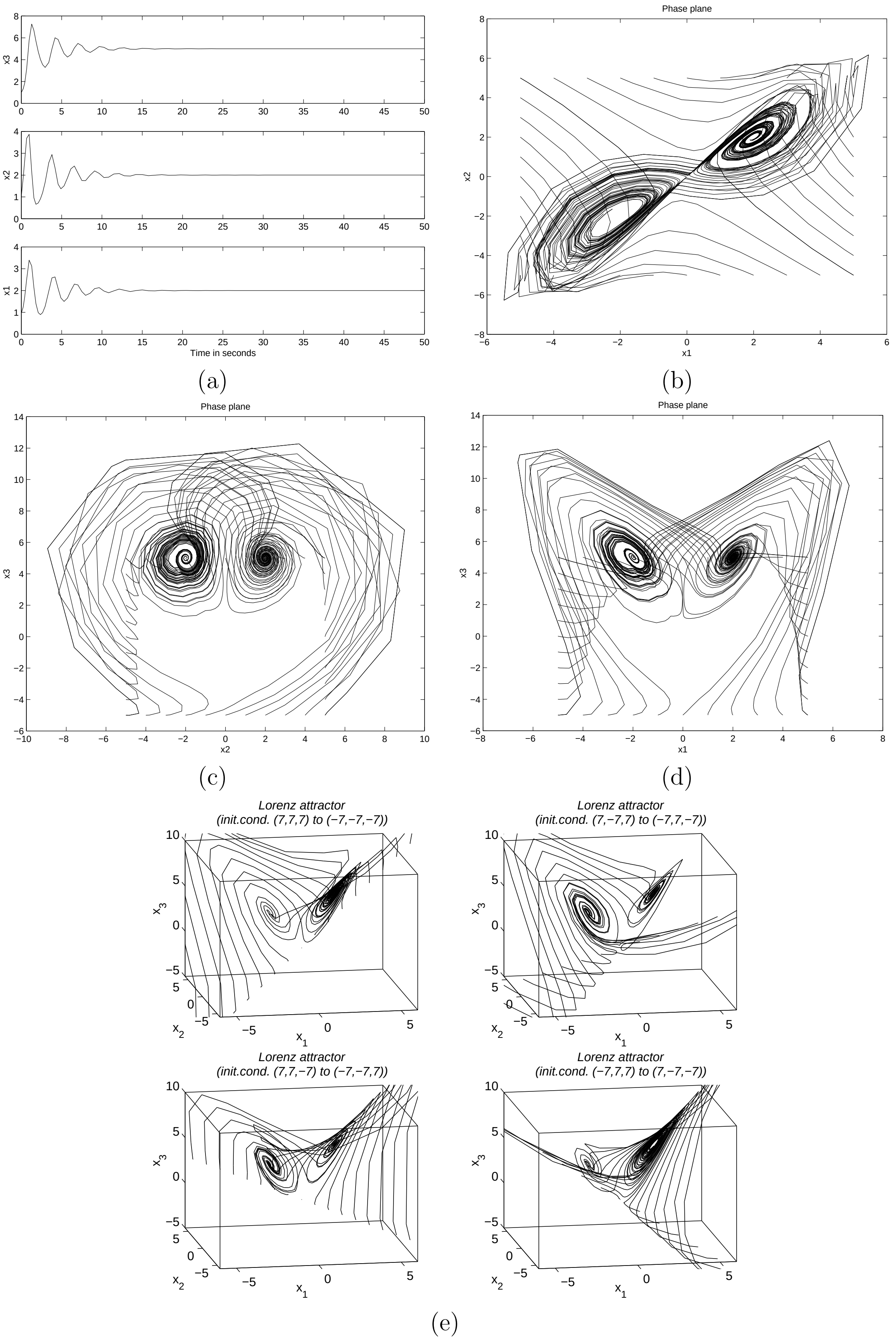


Figure 13: *(a) State variables of a Lorenz attractor ($\sigma = 3$, $\lambda = 5$, $b = 1$), (a) State variables, (b) State plane x_1 v.s. x_2, (c) State plane x_2 v.s. x_3, (d) State plane x_1 v.s. x_3, (e) State space* $u = -\mathrm{sgn}\,(\mathrm{x}_1 + \mathrm{x}_2 + \mathrm{x}_3)$

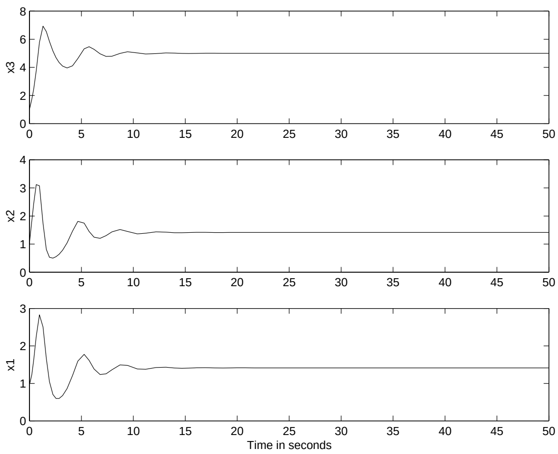
Let $K = -3$ and obtain

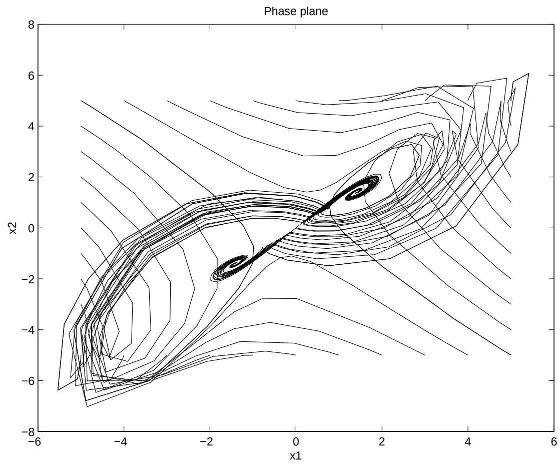
$$u = -3\,\mathrm{sgn}\,(\mathrm{x}_1 + \mathrm{x}_2 + \mathrm{x}_3)\,. \tag{35}$$

The conditions of the system was

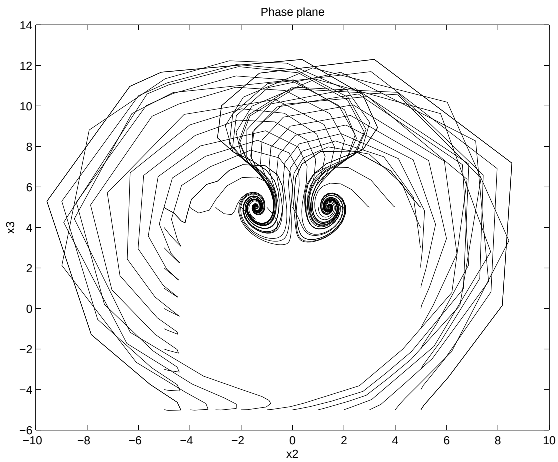
$$\sigma = 3, \quad \lambda = 5, \quad b = 1,$$

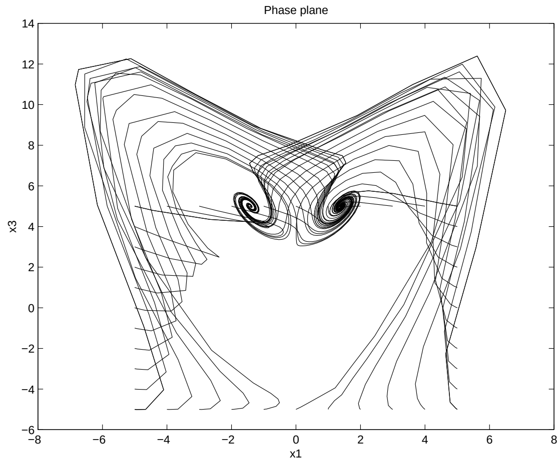
(Equation 29 is satisfied) and the initial conditions used were the same as those shown in page 15. The result is show in Figure 14. Figure 14 has got $t_{CPU} = 26.25$ seconds and its file is 2.38 MB in size.



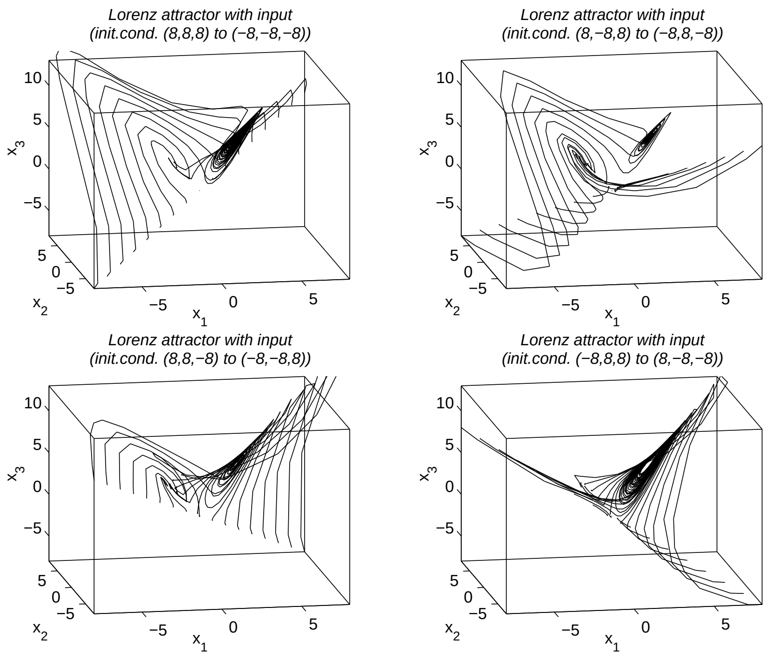


(a) (b)





(c) (d)



(e)

Figure 14: *Lorenz attractor ($\sigma = 3$, $\lambda = 5$, $b = 1$), $t_{simulation} = 50$ sec) (a) State variables, (b) State plane x_1 v.s. x_2, (c) State plane x_2 v.s. x_3,(d) State plane x_1 v.s. x_3,(e) State space.* $u = -3\,\mathrm{sgn}\,(\mathrm{x_1 + x_2 + x_3})$

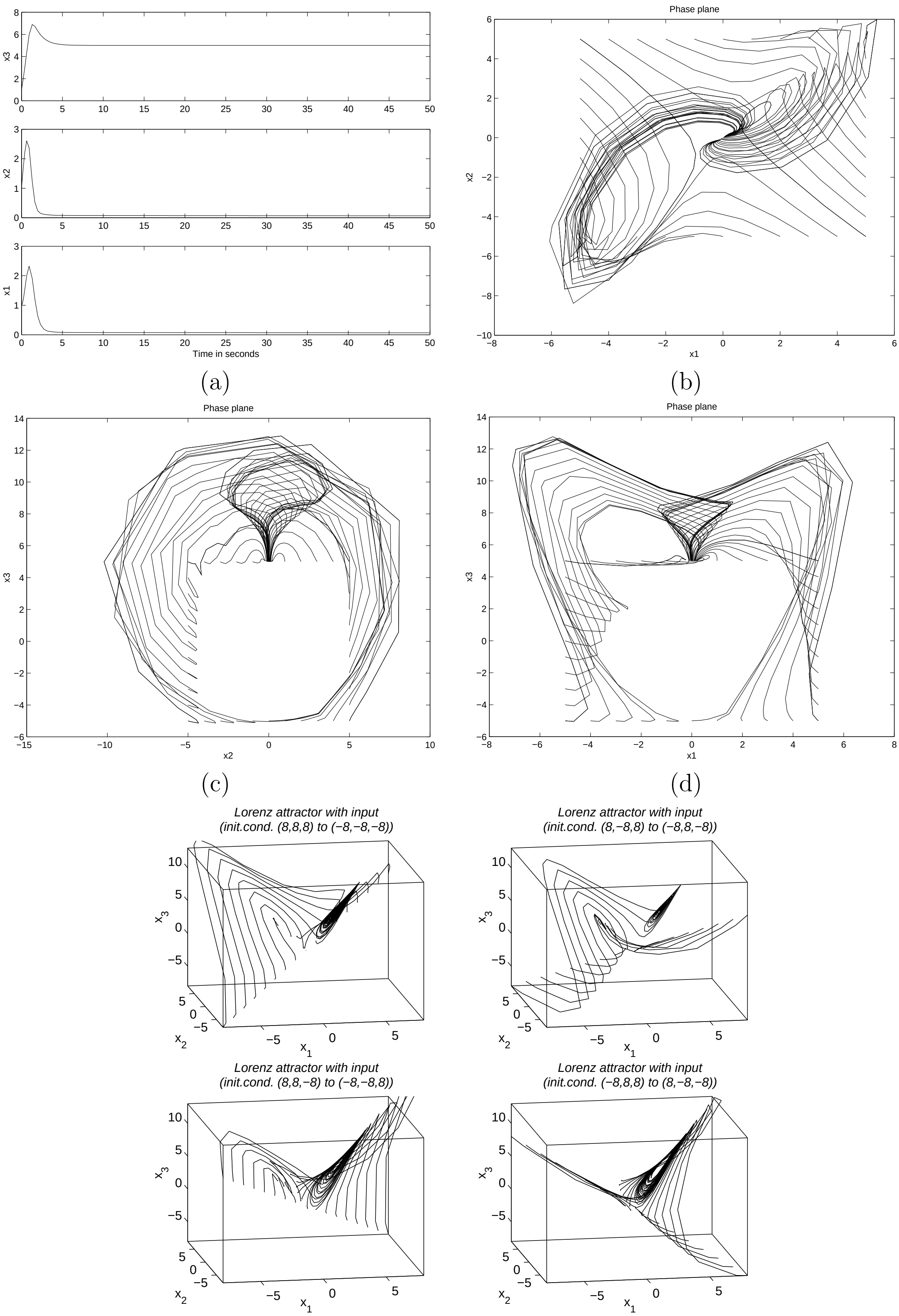
Let $K = -5$ and obtain

$$u = -5\ \mathrm{sgn}\left(\mathrm{x}_1 + \mathrm{x}_2 + \mathrm{x}_3\right). \tag{36}$$

The conditions of the system was

$$\sigma = 3, \quad \lambda = 5, \quad b = 1,$$

(Equation 29 is satisfied) and the initial conditions used were the same to those shown in page 15. Figure 15 shows the result. The file size of Figure 15 (e) is 2.38 MB, and it took $t_{CPU} = 26.25$ seconds.



(a) (b) (c) (d) (e)

Figure 15: *(a) State variables of a Lorenz attractor ($\sigma = 3$, $\lambda = 5$, $b = 1$), (b) State plane x_1 v.s. x_2 , (c) State plane x_2 v.s. x_3, (d) State plane x_1 v.s. x_3, (e) State space.* $u = -5\,\mathrm{sgn}\,(\mathrm{x}_1 + \mathrm{x}_2 + \mathrm{x}_3)$

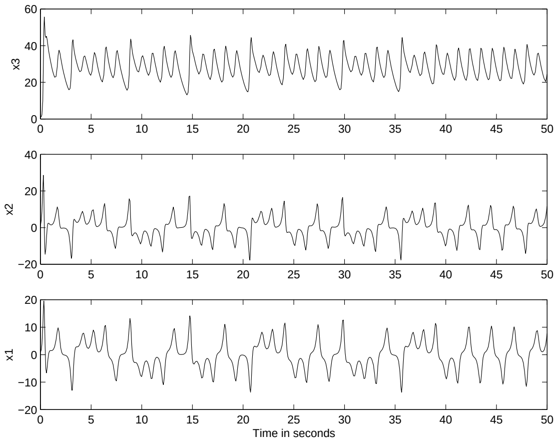
Figure 16 shows the result when the system had

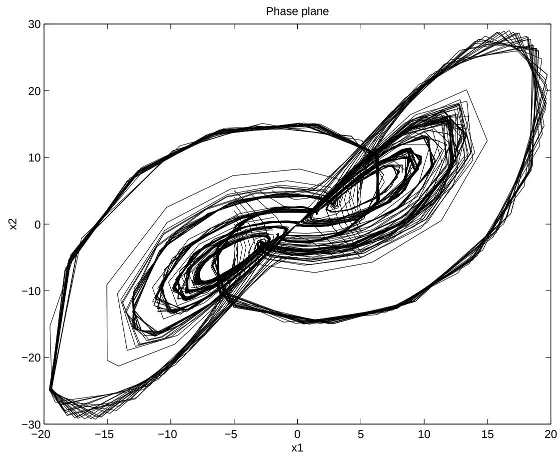
$$\sigma = 9, \quad \lambda = 30, \quad b = 1,$$

(ie. the conditions of Equation 29 was not satisfied) simulated with the same set of initial conditions to those shown in page 15. The input was taken as

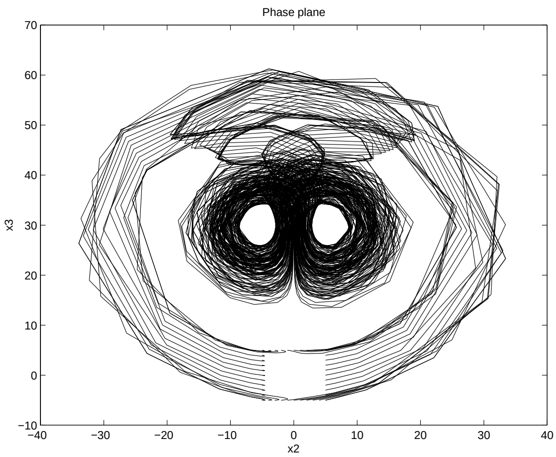
$$u = -\mathrm{sgn}\left(\mathrm{x}_1 + \mathrm{x}_2 + \mathrm{x}_3\right).$$

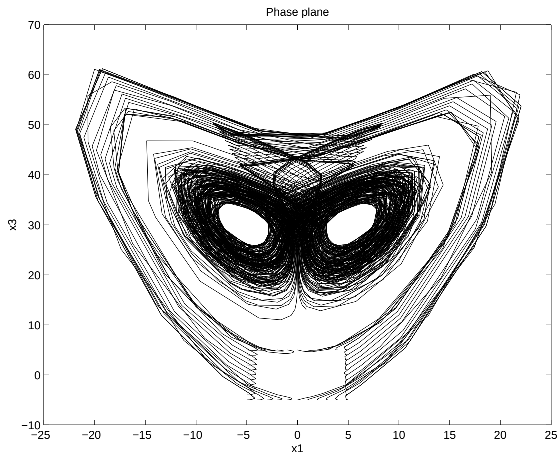
For Figure 16 (e) $t_{CPU} = 2.59$ seconds, and file size = 203.62 KB.



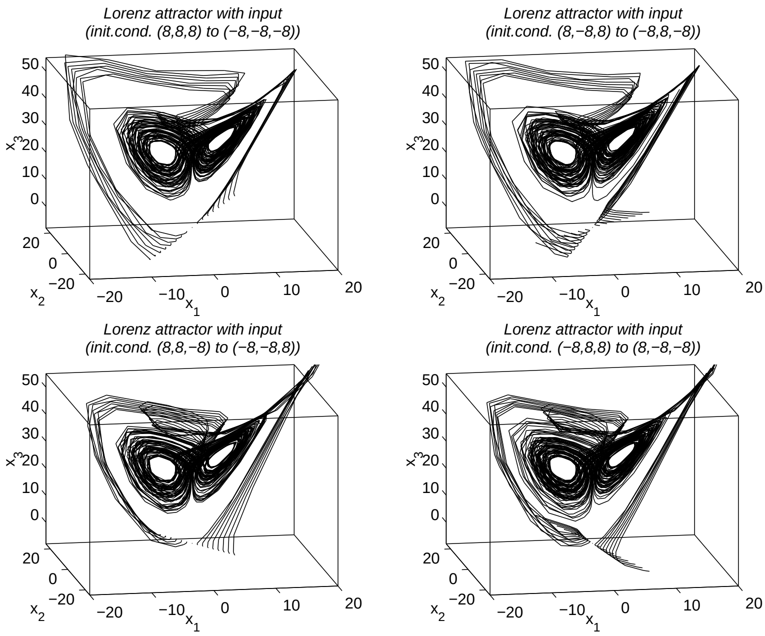


(a) (b)





(c) (d)



(e)

Figure 16: *A Lorenz attractor ($\sigma = 9$, $\lambda = 30$, $b = 1$), (a) State variables, (b) State plane x_1 v.s. x_2, (c) State plane x_2 v.s. x_3, (d) State plane x_1 v.s. x_3, (e),* $u = -\mathrm{sgn}\,(\mathrm{x}_1 + \mathrm{x}_2 + \mathrm{x}_3)$

Let the input be

$$u = -5\,\mathrm{sgn}\left(\mathrm{x}_1 + \mathrm{x}_2 + \mathrm{x}_3\right),$$

and the system has

$$\sigma = 9, \quad \lambda = 30, \quad b = 1,$$

so that the conditions of Equation 29 was not satisfied. Then with the set of initial conditions specified in page 15 simulated and obtained the result of Figure 17. Figure 17 has $t_{CPU} = 2.62$ seconds, and file size = 209.80 KB.

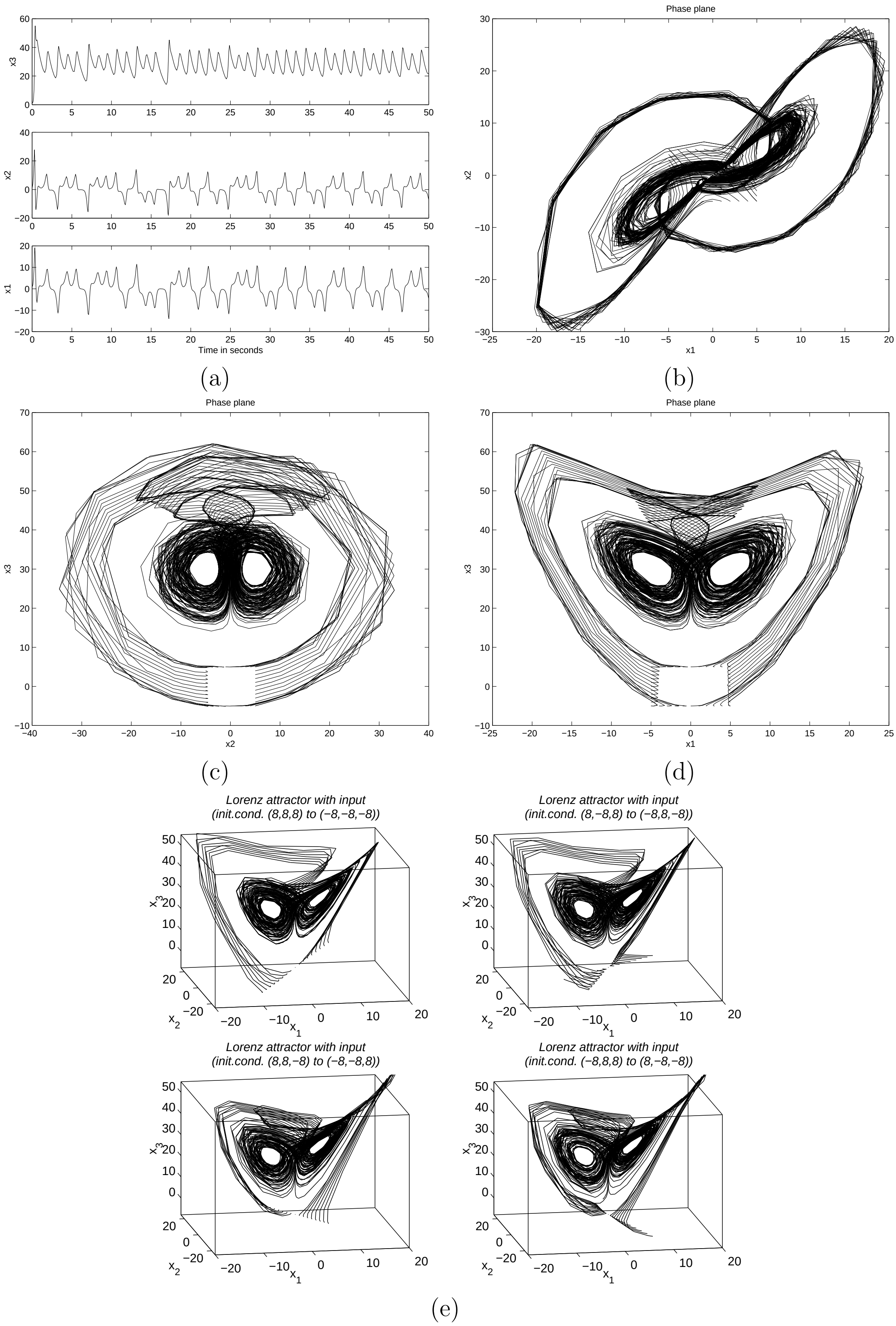


Figure 17: *A Lorenz attractor ($\sigma = 9$, $\lambda = 30$, $b = 1$), (a) State variables, (b) State plane x_1 v.s. x_2, (c) State plane x_2 v.s. x_3, (d) State plane x_1 v.s. x_3, (e),* $u = -5\mathrm{sgn}\left(\mathrm{x}_1 + \mathrm{x}_2 + \mathrm{x}_3\right)$

Feeding back $u = k\,\mathrm{sgn}(\mathrm{x}_1 + \mathrm{x}_2 + \mathrm{x}_3)$ to the system with

$$k > 0$$

led to the response shown in Figure 18. Here the two equilibrium points away from the origin are not stable. Figure 18 was from a simulation with

$$u = 3\,\mathrm{sgn}\,(\mathrm{x}_1 + \mathrm{x}_2 + \mathrm{x}_3)$$

as an input. The system had

$$\sigma = 9, \quad \lambda = 30, \quad b = 1,$$

so that the conditions of Equation 29 was not satisfied. Then simulated with the set of initial conditions the same with those given in page 15. Figure 18 (e) has $t_{CPU} = 2.64$ seconds, and file size = 206.55 KB.

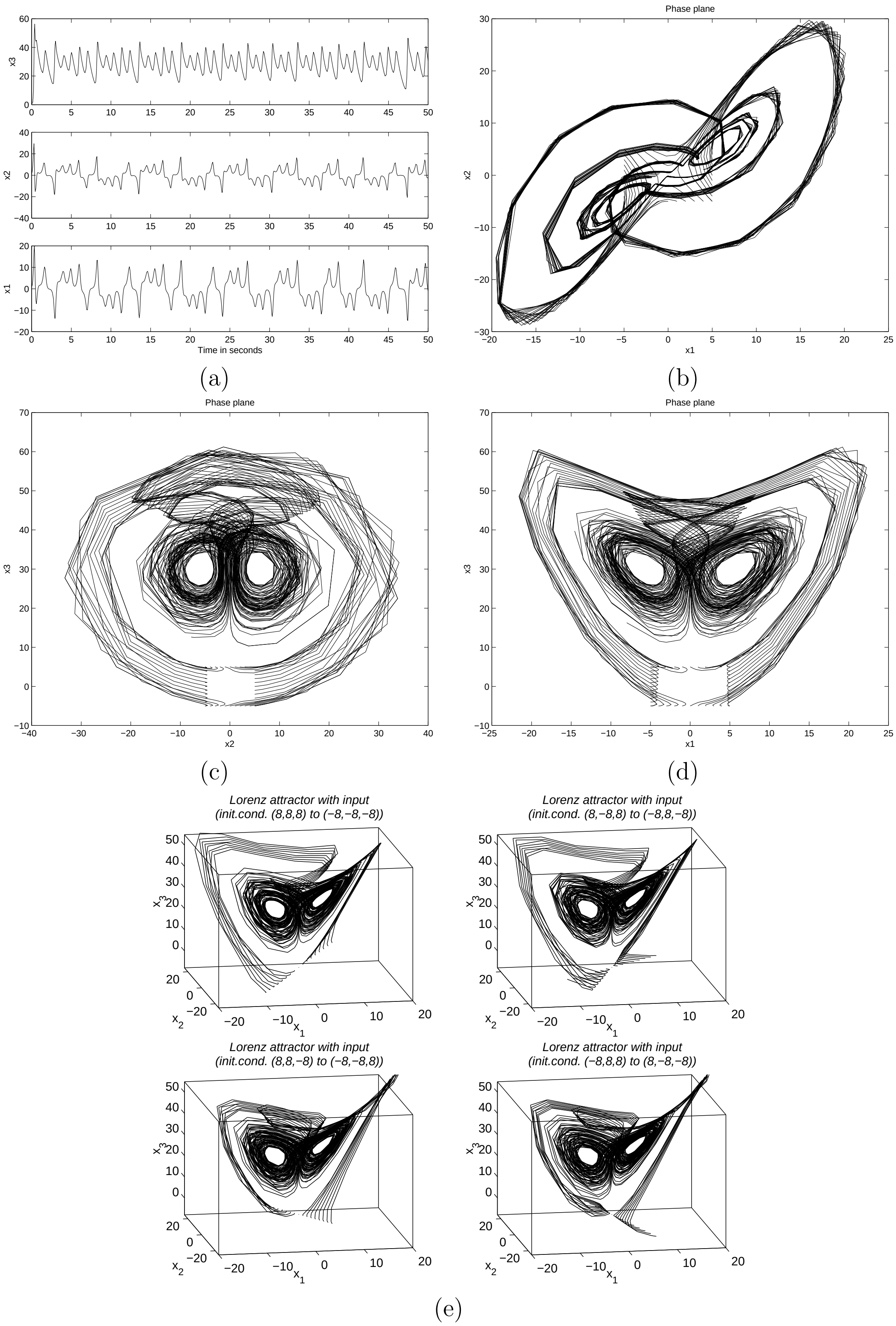


Figure 18: *A Lorenz attractor ($\sigma = 9$, $\lambda = 30$, $b = 1$), (a) State variables, (b) State plane x_1 v.s. x_2, (c) State plane x_2 v.s. x_3, (d) State plane x_1 v.s. x_3, (e),* $u = 3\,\mathrm{sgn}\,(\mathrm{x}_1 + \mathrm{x}_2 + \mathrm{x}_3)$

Discussion

Synchronous machine

11^{th} **May 1998**
Consider a model of a synchronous electrical machine [Coo86] used in power system engineering.

$$H\ddot{\delta} = P_m - P_e \sin\delta \tag{37}$$

where δ is the rotor angle relative to a reference frame, H is the inertia constant, P_m is the mechanical power supplied, and P_e is the maximum electrical power which can be generated. This equation can be put into state-space form as

$$\dot{x_1} = x_2 \tag{38}$$

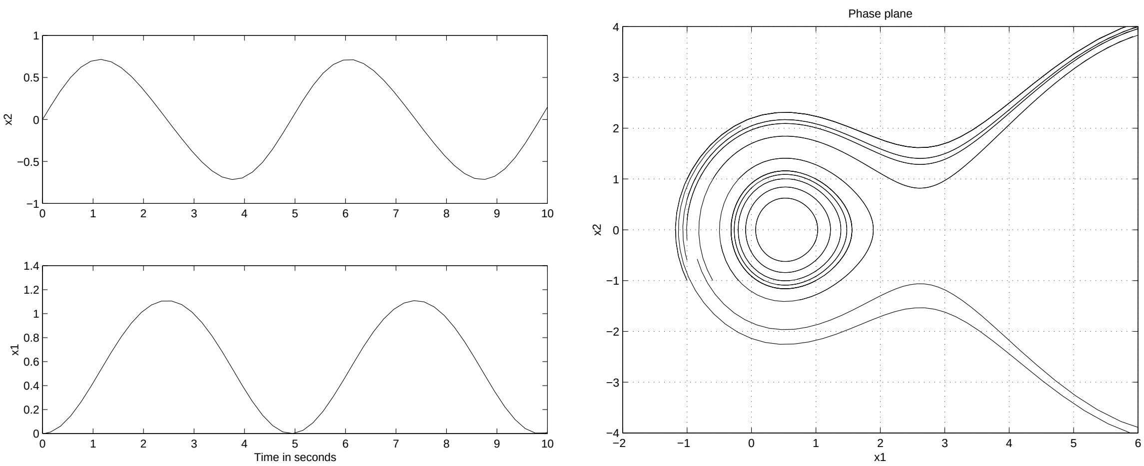
$$\dot{x_2} = \frac{u - P_e \sin x_1}{H} \tag{39}$$

$$y = P_e \sin x_1 \tag{40}$$

where $x_1 = \delta$, $x_2 = \dot{\delta}$ and $u = P_m$.

Figure 19 shows the state variables and the phase plane of this machine respectively when

$$H = 1, \quad P_m = 1, \quad P_e = 2.$$



(a) (b)

Figure 19: *A synchronous machine ($H = 1$, $P_m = 1$, $P_e = 2$), (a) State variables, and*

Unless the mechanical power is within the bounds defined by the maximum electrical power the system can not operate in equlibrium. Figure's 20 (a) and (b) shows the state variables and the phase plane of this machine respectively when

$$H = 1, \quad P_m = 3, \quad P_e = 2,$$

that is $P_m > P_e$.

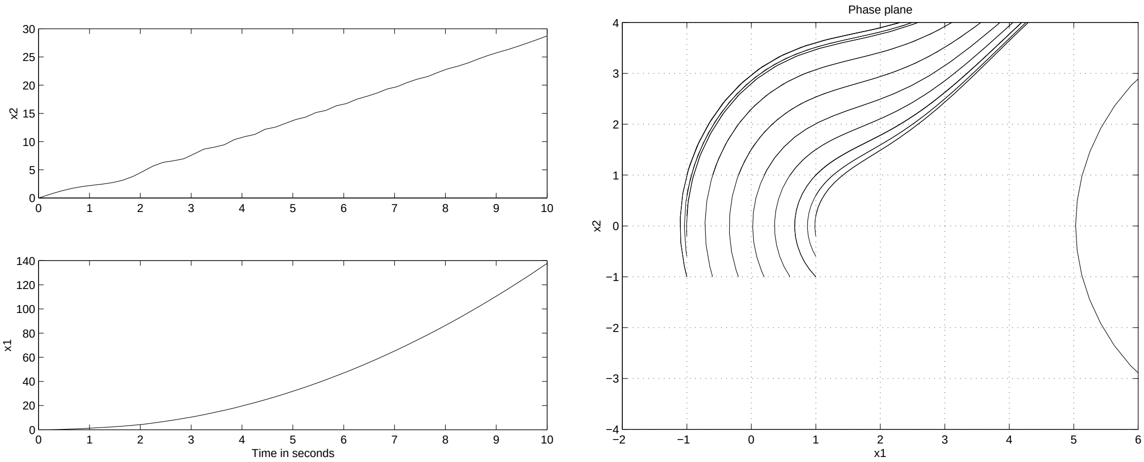


Figure 20: *A synchronous machine ($H = 1$, $P_m = 3$, $P_e = 2$), (a) State variables, and (b) Phase plane.*

When

$$P_m < 0$$

the machine takes power from the bus and acts as a motor instead of a generator. Figure 21 (a) and (b) shows the state variables and the phase plane of this machine respectively. Here

$$H = 1, \quad P_m = -1, \quad P_e = 2.$$

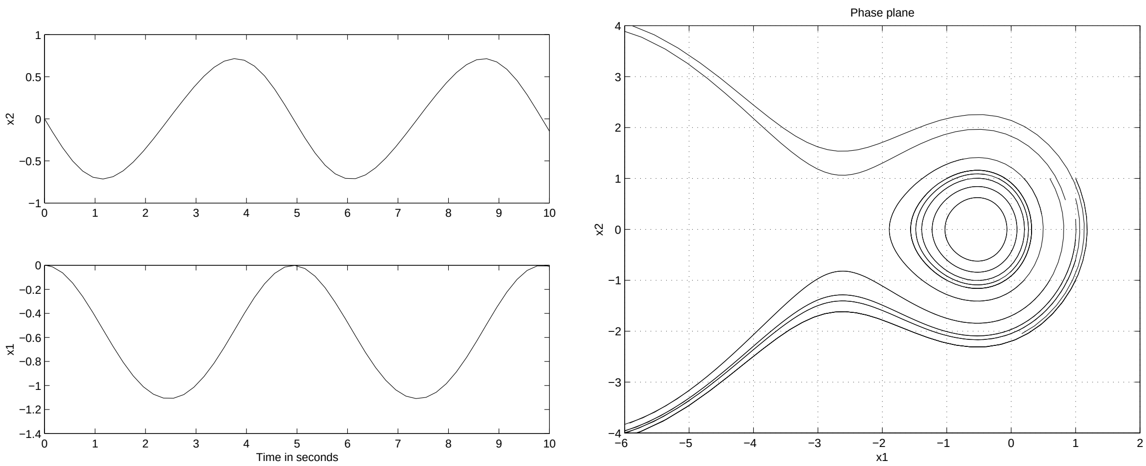


Figure 21: *A synchronous machine ($H = 1$, $P_m = -1$, $P_e = 2$), (a) State variables, and (b) Phase plane.*

If

$$|P_m| > |P_e|$$

the synchronous motor will not operate at equilibrium as shown in Figure 22.

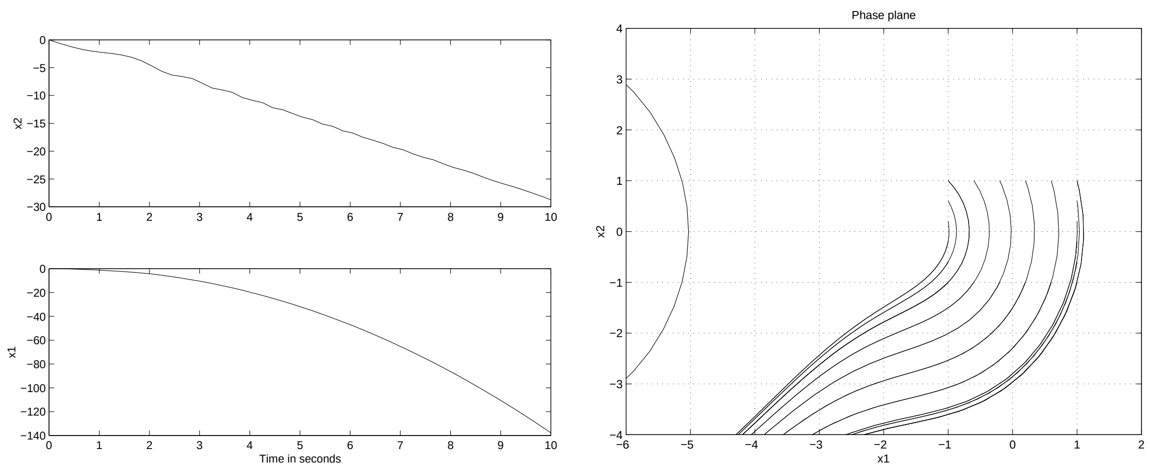


Figure 22: *A synchronous machine ($H = 1$, $P_m = -3$, $P_e = 2$), (a) State variables, and (b) Phase plane.*

With a signum function feedback

Fed the sign function of state variables back to input of the machine and obtained Figure 23. Here

$$H = 1, \quad P_m = 1, \quad P_e = 2.$$

(a generator)

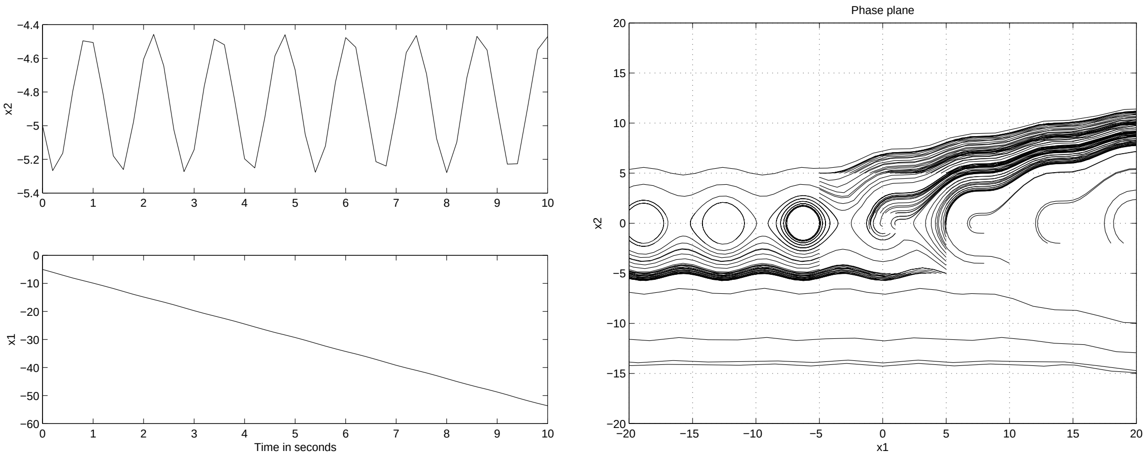


Figure 23: *A synchronous machine ($H = 1$, $P_m = 1$, $P_e = 2$), (a) State variables, and (b) Phase plane, with a sign function feedback.*

Figure's 24 (a) and (b) shows the results, also with a sign function in feedback, but this time

$$P_m < 0.$$

(a motor) The system had

$$H = 1, \quad P_m = -1, \quad P_e = 2.$$

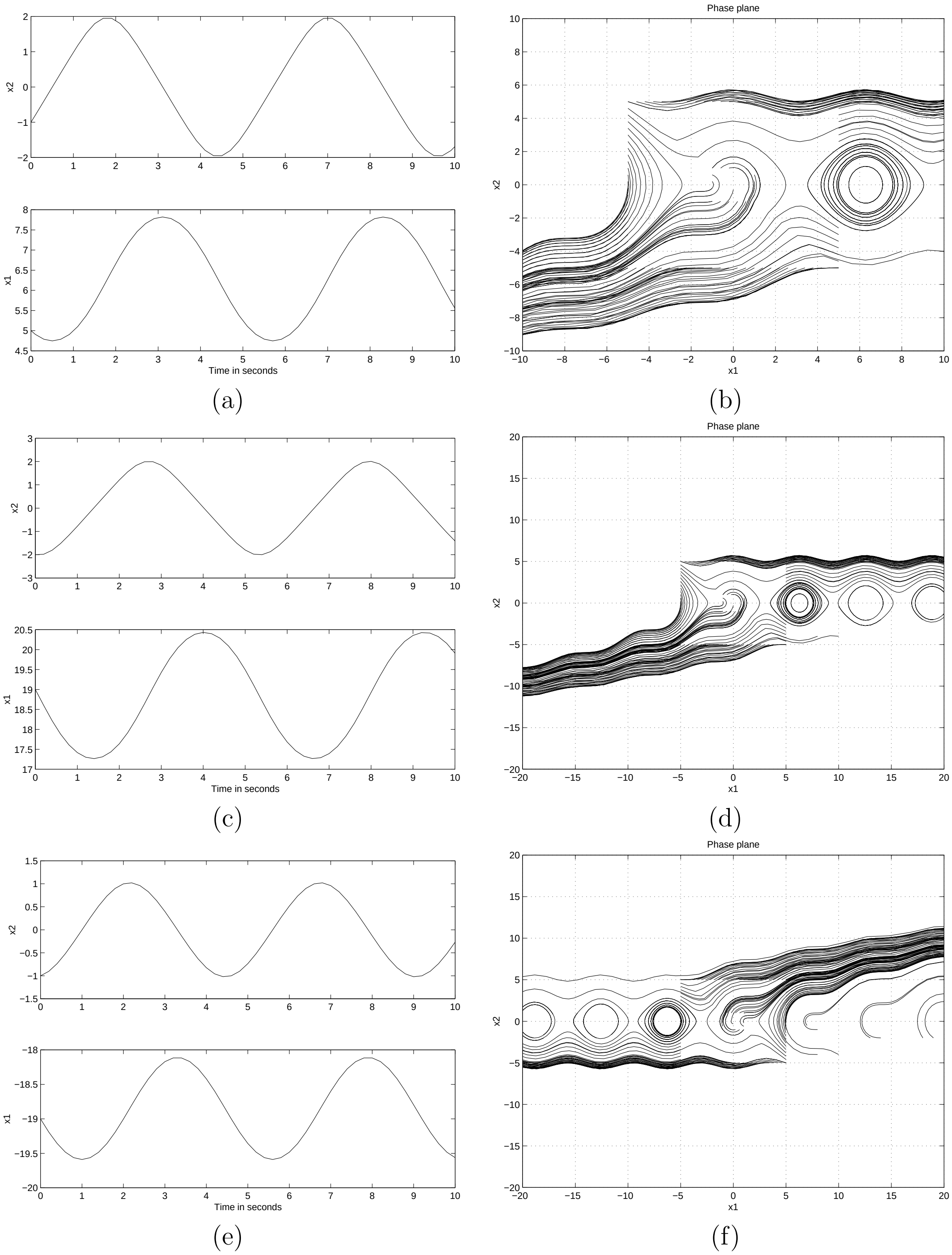


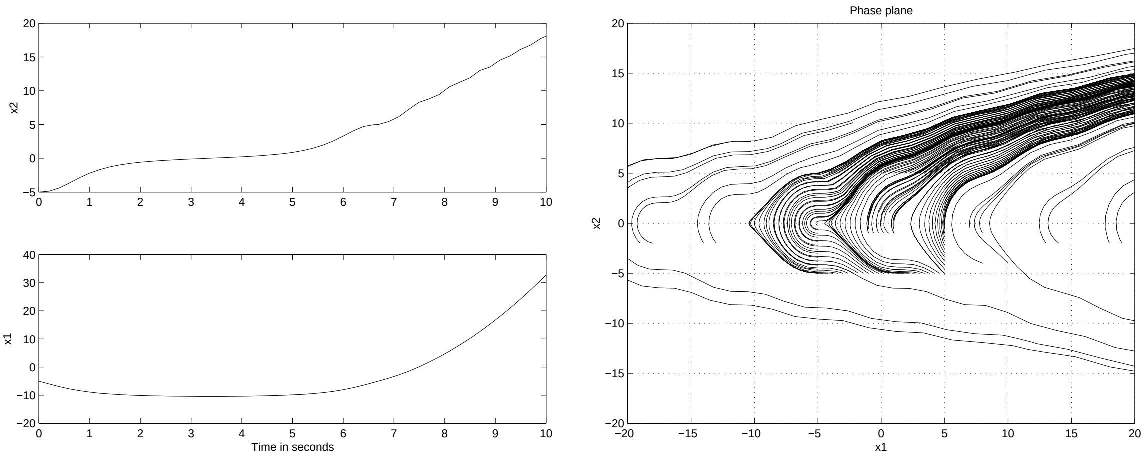
Figure 24: *A synchronous motor ($H = 1$, $P_m = -1$, $P_e = 2$), (a), (c), (e) State variable plots, and (b), (d), (f) are Phase planes, with a sign function feedback.*

When $|P_m| > |P_e|$

Again, P_m must satisfy the condition

$$|P_m| > |P_e|$$

or the results similar to that shown in Figure 25 (a) and (b) will be obtained.



(a) (b)

Figure 25: *A synchronous motor ($H = 1$, $P_m = 3$, $P_e = 2$), (a) State variable plots, and (b) Phase planes, with a sign function feedback.*

Next figure, Figure 26 shows the phase plane of a condition similar to that of Figure 25 but where $P_e = -2$ instead of 2.

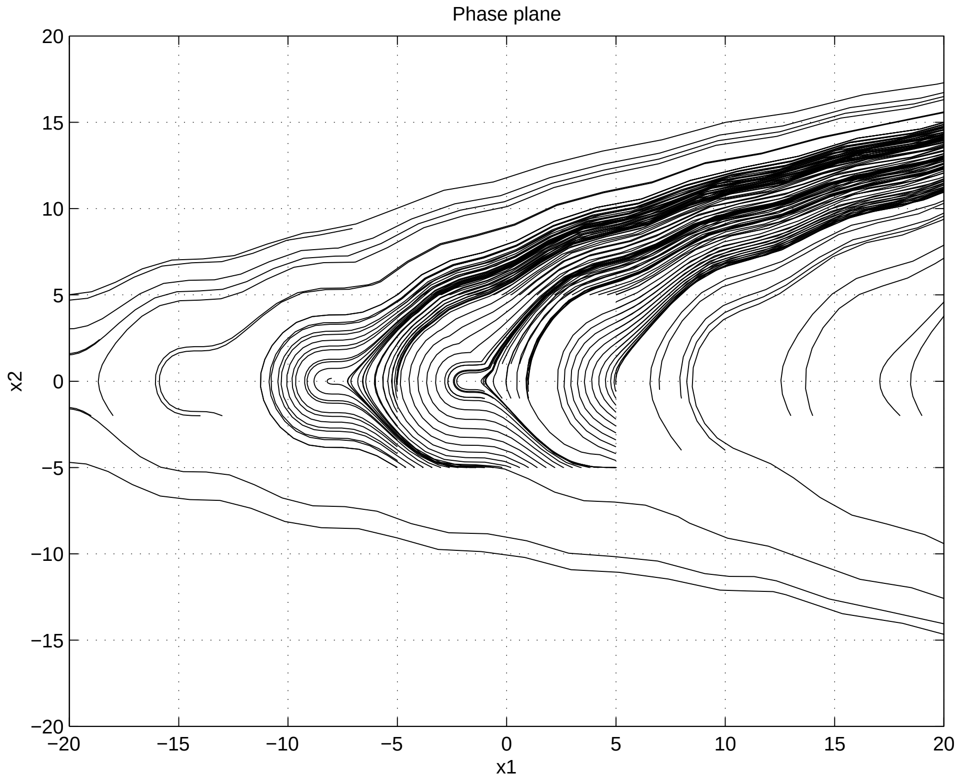
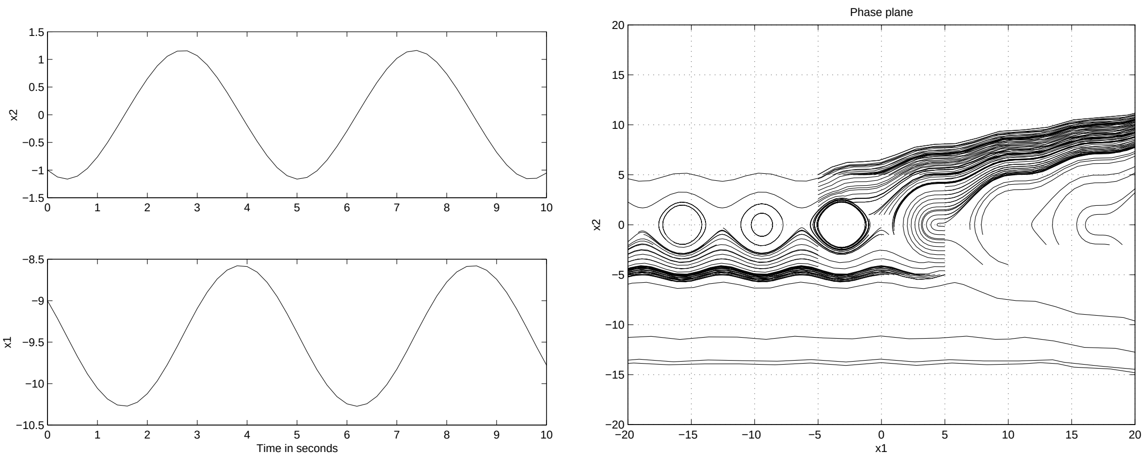


Figure 26: *A synchronous motor ($H = 1$, $P_m = 3$, $P_e = -2$), (a) State variable plots, and (b) Phase planes, with a sign function feedback.*

With $P_m > 0$ but $P_e < 0$

Figure 27 shows the result when conditions are similar to those of Figure 23 except that this time $P_e = -2$ instead of 2.



(a) (b)

Figure 27: *A synchronous motor ($H = 1$, $P_m = 1$, $P_e = -2$), (a) State variable plots, and (b) Phase planes, with a sign function feedback.*

Figure 28 (b) is similar to Figure 28 (a) except that here there is no feedback of sign function. By comparing the two figures one can see that about half of the equilibriums were lost when a sign function FB was introduced. The parameters used in simulating Figure 28 were

$$H = 1, \quad P_m = 1, \quad P_e = 2.$$

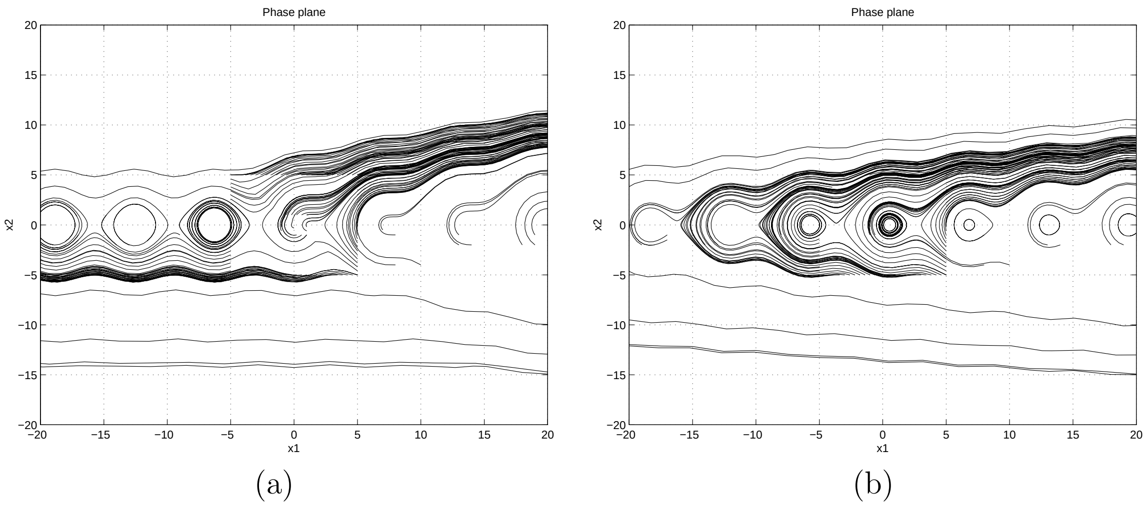


Figure 28: *A a synchronous machine ($H = 1$, $P_m = 1$, $P_e = 2$). (a) with a sign function FB and, (b) without a sign function FB.*

Figure 29 shows the result when

$$H = 1, \quad P_m = 1, \quad P_e = 2,$$

and

$$u = -5\mathrm{sgn}(\mathrm{x}_1 + \mathrm{x}_2)$$

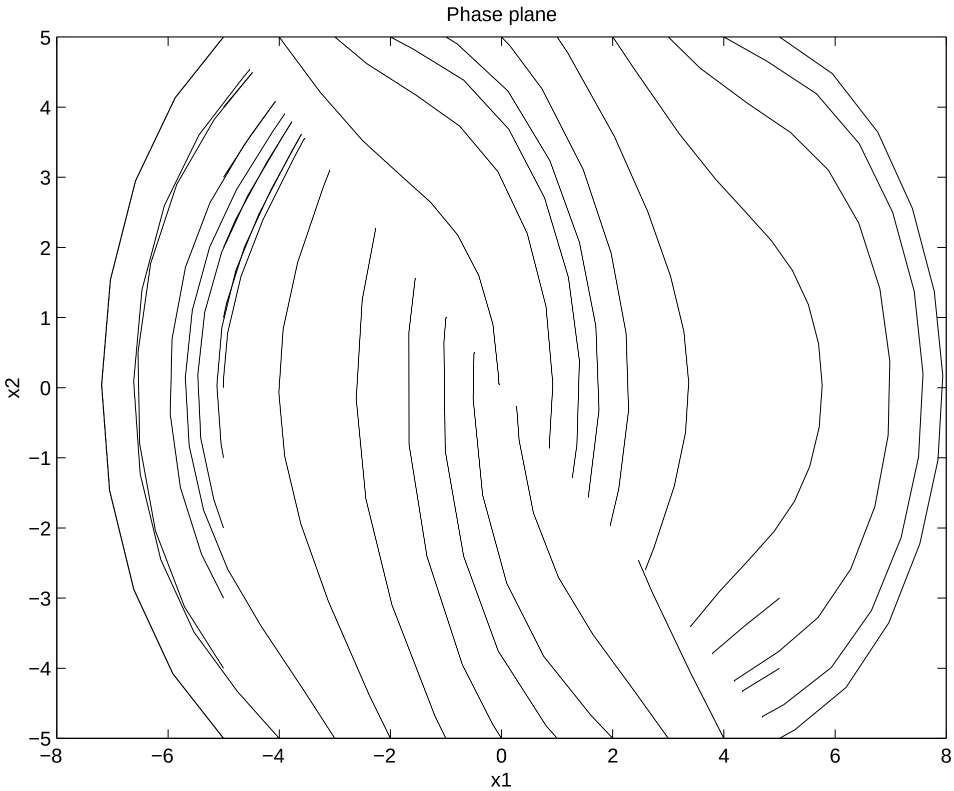


Figure 29: *Phase plane of a synchronous machine ($H = 1$, $P_m = 1$, $P_e = 2$).* $u = -5\mathrm{sgn}(\mathrm{x}_1 + \mathrm{x}_2)$

sgn (P_m)	sgn $(\P_e(0))$	sgn$(\delta(0))$	Figure #
v+	$+$	$+$	Figure 30
$+$	$+$	$-$	Figure 31
$+$	$-$	$+$	Figure 32
$+$	$-$	$-$	Figure 33
$-$	$+$	$+$	Figure 34
$-$	$+$	$-$	Figure 35
$-$	$-$	$+$	Figure 36
$-$	$-$	$-$	Figure 37

Table 1: *Sign table for the investigation to study the effect of initial conditions on response*

Initial conditions #	$\lvert P_m\rvert$	$\lvert P_e(0)\rvert$	$\lvert\delta(0)\rvert$
# 1	sgn $(P_m)\cdot 1$	sgn $(P_e(0))\cdot 1.5$	sgn $(\delta(0))\cdot \frac{\pi}{2}$
# 2	sgn $(P_m)\cdot 1.5$	sgn $(P_e(0))\cdot 1$	sgn $(\delta(0))\cdot \frac{\pi}{2}$
# 3	sgn $(P_m)\cdot 3$	sgn $(P_e(0))\cdot 3$	sgn $(\delta(0))\cdot \frac{3\pi}{4}$
# 4	sgn $(P_m)\cdot 5$	sgn $(P_e(0))\cdot 5$	sgn $(\delta(0))\cdot \frac{\pi}{4}$
# 5	sgn $(P_m)\cdot 0.1$	sgn $(P_e(0))\cdot 0.1$	sgn $(\delta(0))\cdot \frac{\pi}{6}$

Table 2: *The initial conditons to study the effect of initial conditions on response*

Three-dimensional model of synchronous machine

Consider the three-dimensional model of the synchronous machine (see also [Tiy98]) described by

$$\dot{x_1} = x_2 \tag{41}$$

$$\dot{x_2} = \frac{P_m}{H} - \frac{D}{H}x_2 - \frac{\eta_1}{H}x_3 \sin x_1 \tag{42}$$

$$\dot{x_3} = -\frac{\eta_2}{\tau}x_3 + \frac{\eta_3}{\tau}\cos x_1 + \frac{E_{FD}}{\tau}, \tag{43}$$

where

$$x_1 = \delta(t), \quad x_2 = \dot{\delta}(t), \quad x_3 = P_e(t).$$

The following study follows the initial conditions and parameter set described by Table 1 and Table 2. The rest of the parameters are

$$P_m = 1, H = 0.03, \eta_1 = 2.2, \eta_2 = 2.9, \eta_3 = 1.7, \tau = 8, E_{FD} = 1.45, \frac{D}{H} = -6,$$

and another initial condition still to be mentioned is

$$\dot{\delta}(0) = 0.$$

The following are the figures mentioned in Table 1.

The result from Figure 30 to Figure 37 can be summarized as follows. The system was unstable (ie. δ increased indefinitely) when

1. $P_m = \pm 5, \quad P_e(0) = \pm 5, \quad \delta(0) = \pm\frac{\pi}{4}$ (every figure from Figure 30 to 37),
2. $P_m = \pm 3, \quad P_e(0) = \pm 3, \quad \delta(0) = \pm\frac{3\pi}{4}$ (every figure from Figure 30 to 37),

3. $P_m = \pm 1, \quad P_e(0) = -1.5, \quad \delta(0) = \pm\frac{\pi}{2}$ (Figure 32, 33, 36, and 37), or
4. $P_m = \pm 1.5, \quad P_e(0) = -1, \quad \delta(0) = \pm\frac{\pi}{2}$ (Figure 32, 33, 36, and 37).

The system was stable (ie. δ did not increase indefinitely) when

1. $P_m = \pm 0.1, \quad P_e(0) = \pm 0.1, \quad \delta(0) = \pm\frac{\pi}{6}$ (every figure from Figure 30 to 37),
2. $P_m = \pm 1, \quad P_e(0) = 1.5, \quad \delta(0) = \pm\frac{\pi}{2}$ (Figure 30, 31, 34, and 35), or
3. $P_m = \pm 1.5, \quad P_e(0) = 1, \quad \delta(0) = \pm\frac{\pi}{2}$ (Figure 30, 31, 34, and 35).

Notice that $P_e > 0$ denotes electrical power produced from the machine, while $P_e < 0$ denotes the opposite, ie. the electrical power injected into the machine. Therefore, to put it another words is that no matter whether the machine is having $P_m > 0$ (mechanical power input into the machine) or $P_m < 0$ (mechanical power produced by the machine), when ever $P_e < 0$ the machine becomes less robust to the initial states of the state variables. Or we can say that the domain of attraction in the phase plane when $P_e < 0$ is smaller when that when $P_e > 0$.

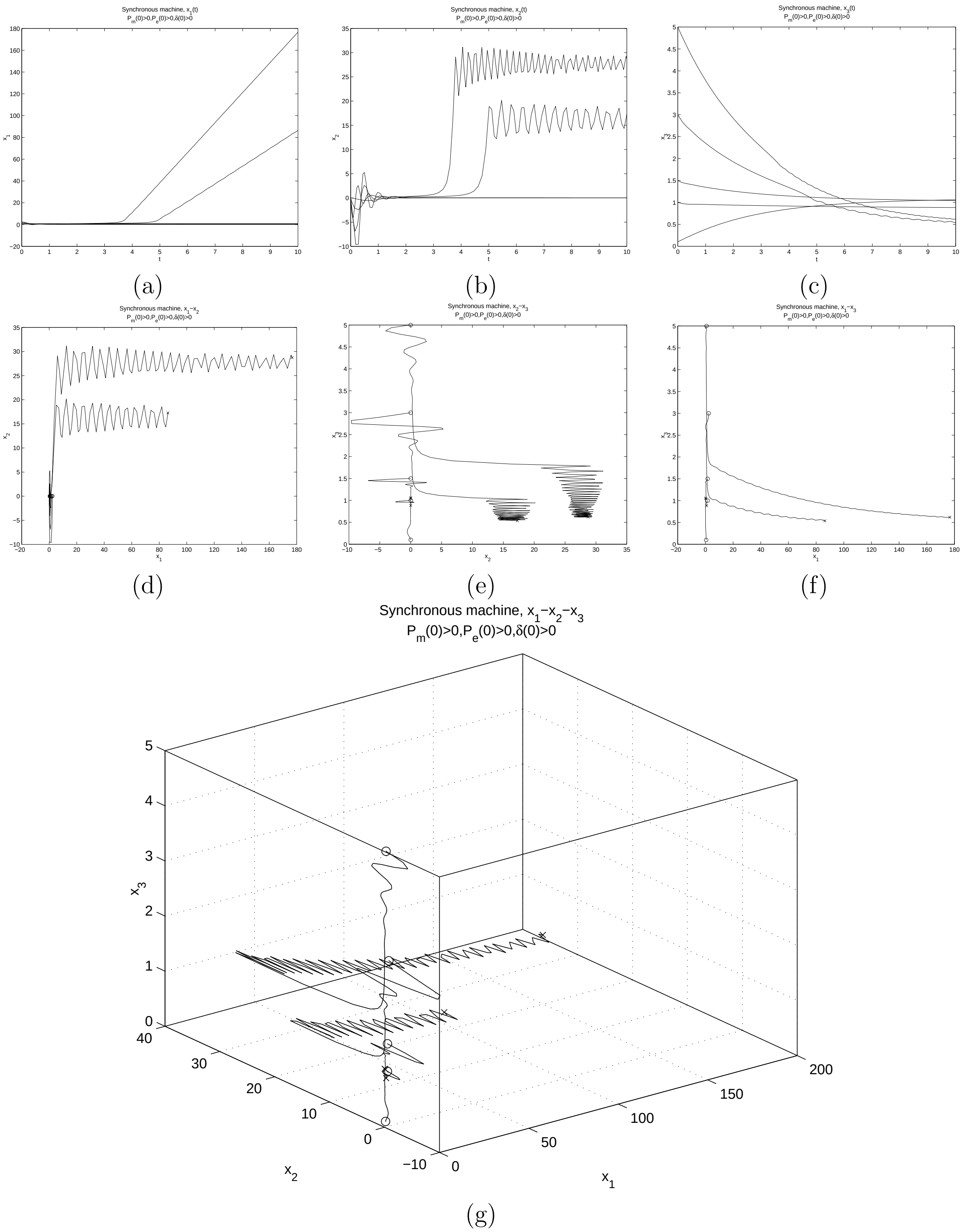


Figure 30: $\delta(0) > 0,\ \dot{\delta}(0) > 0,\ P_e(0) > 0.$

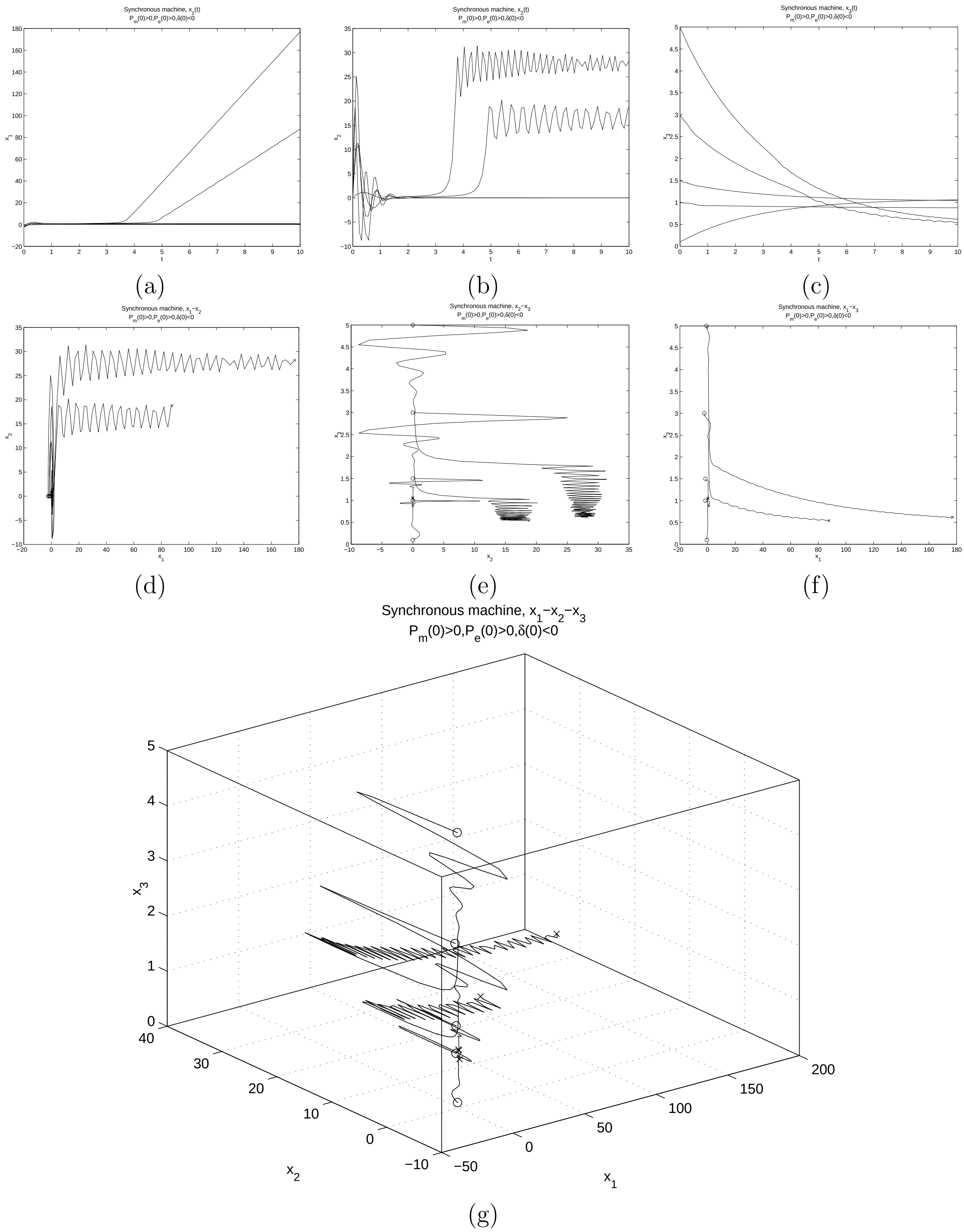


Figure 31: $\delta(0) > 0,\ \dot{\delta}(0) > 0,\ P_e(0) > 0.$

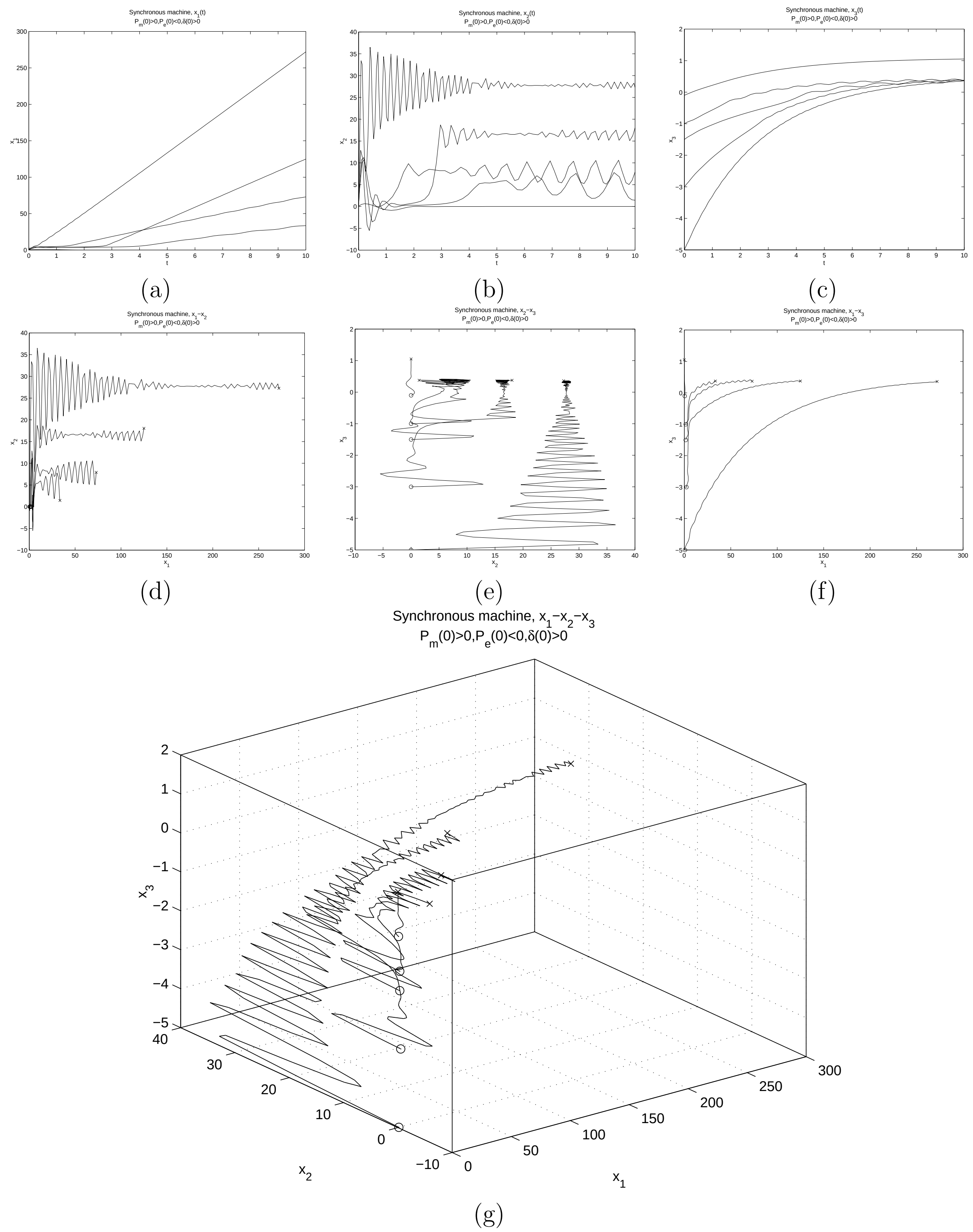


Figure 32: $\delta(0) > 0$, $\dot{\delta}(0) > 0$, $P_e(0) > 0$.

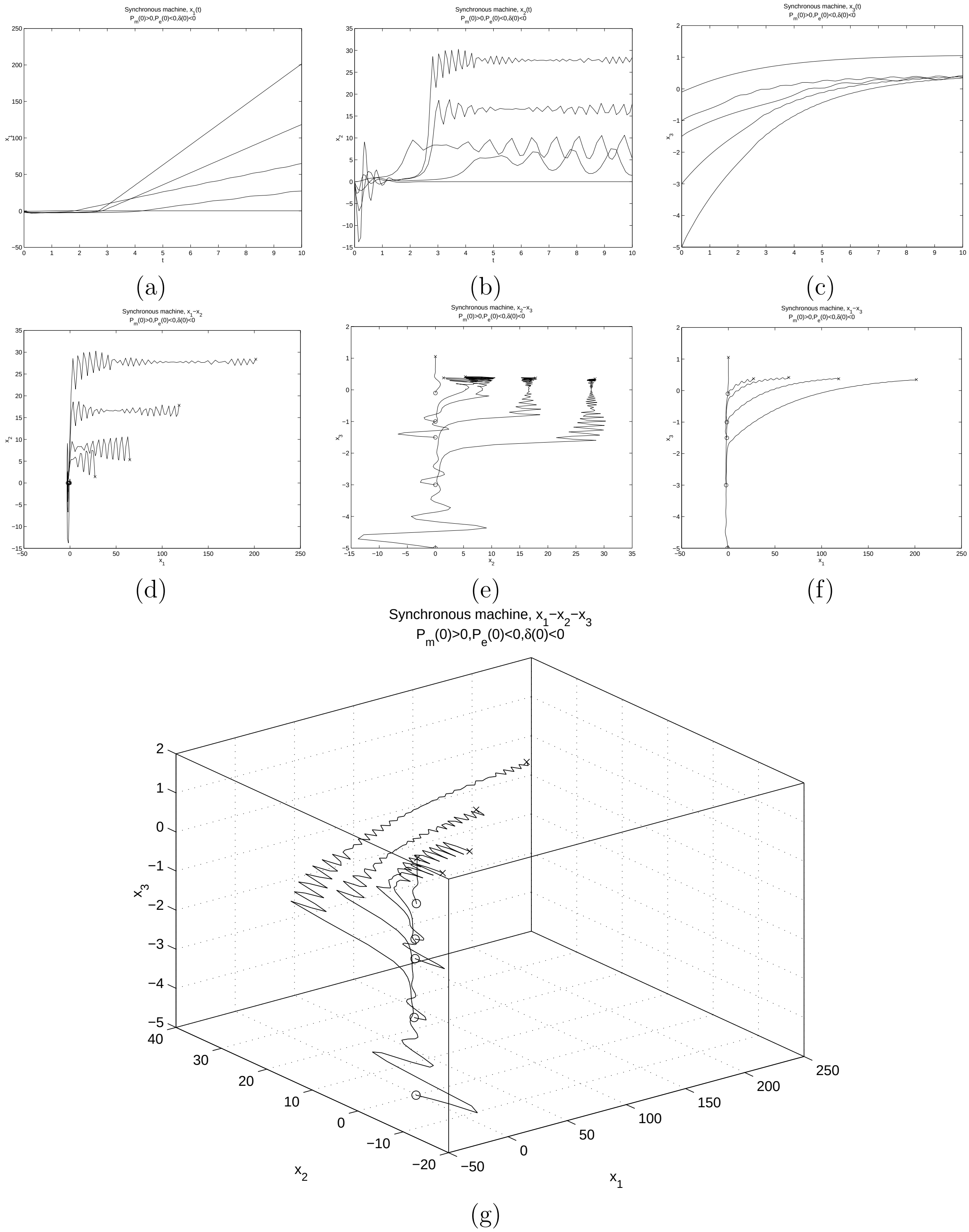


Figure 33: $\delta(0) > 0,\ \dot{\delta}(0) > 0,\ P_e(0) > 0.$

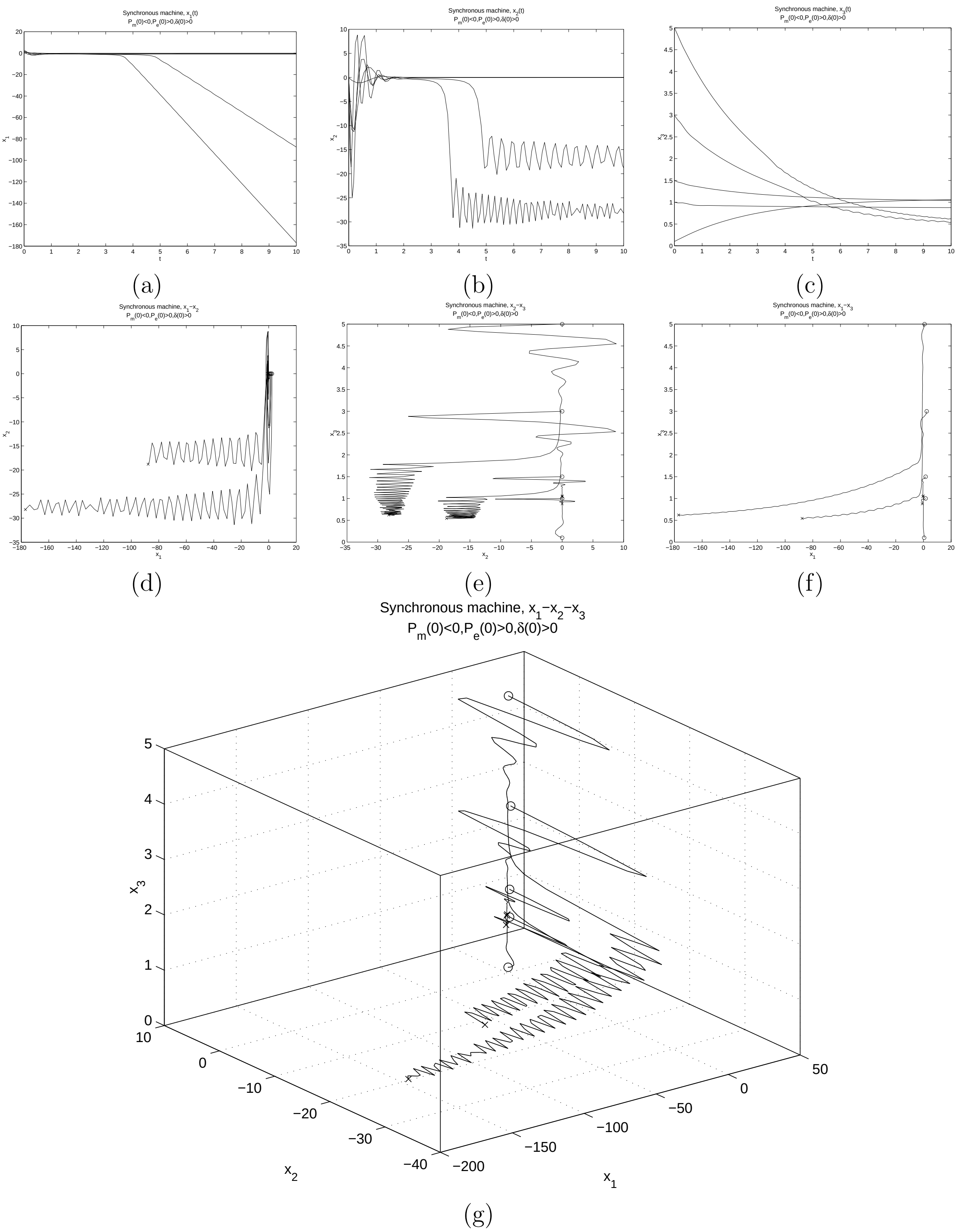
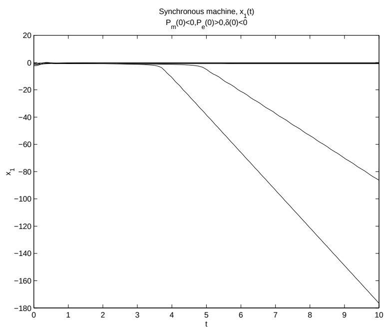
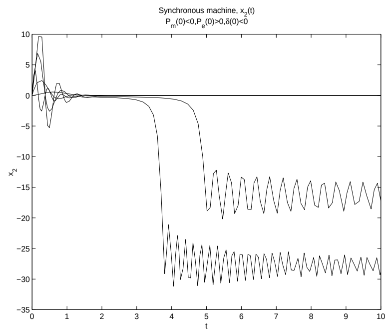


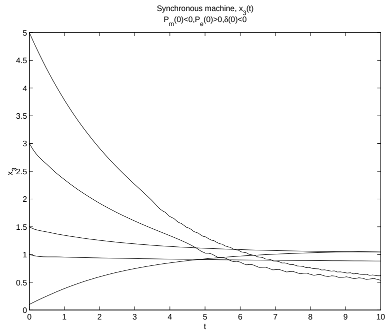
Figure 34: $\delta(0) > 0$, $\dot{\delta}(0) > 0$, $P_e(0) > 0$.



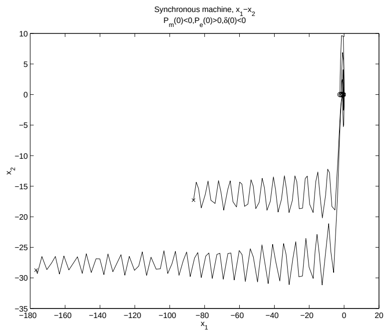
(a)



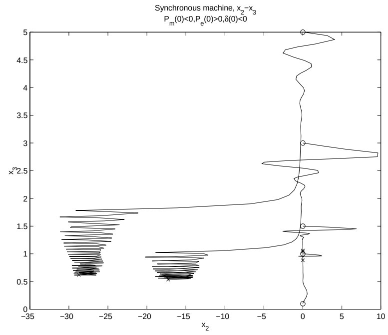
(b)



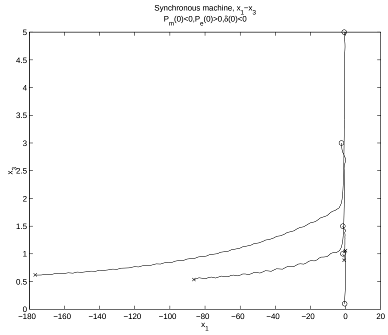
(c)



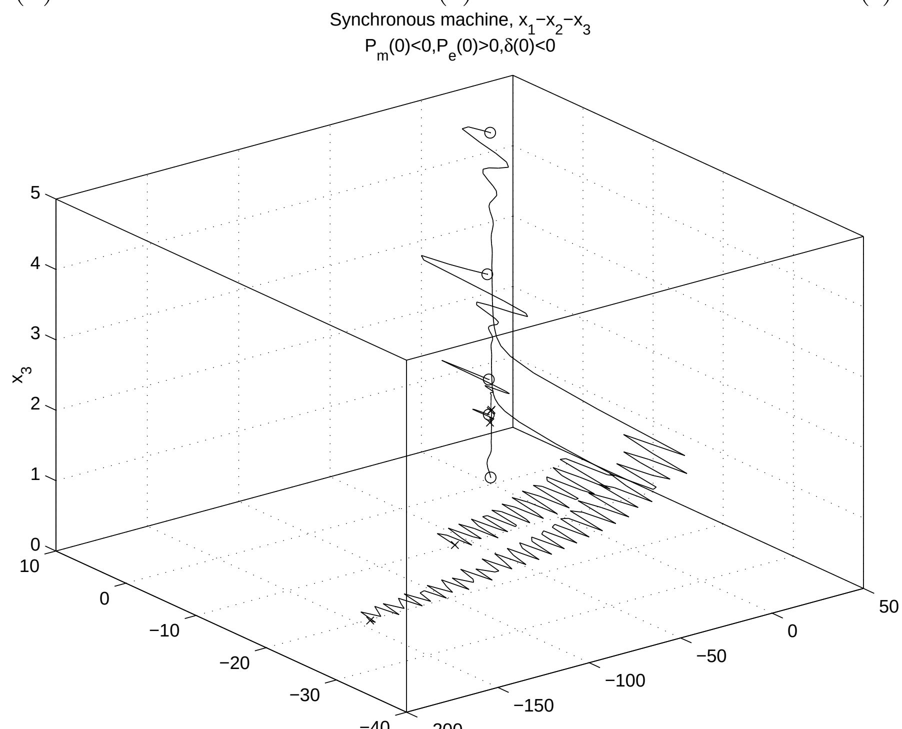
(d)



(e)



(f)



(g)

Figure 35: $\delta(0) > 0,\ \dot{\delta}(0) > 0,\ P_e(0) > 0.$

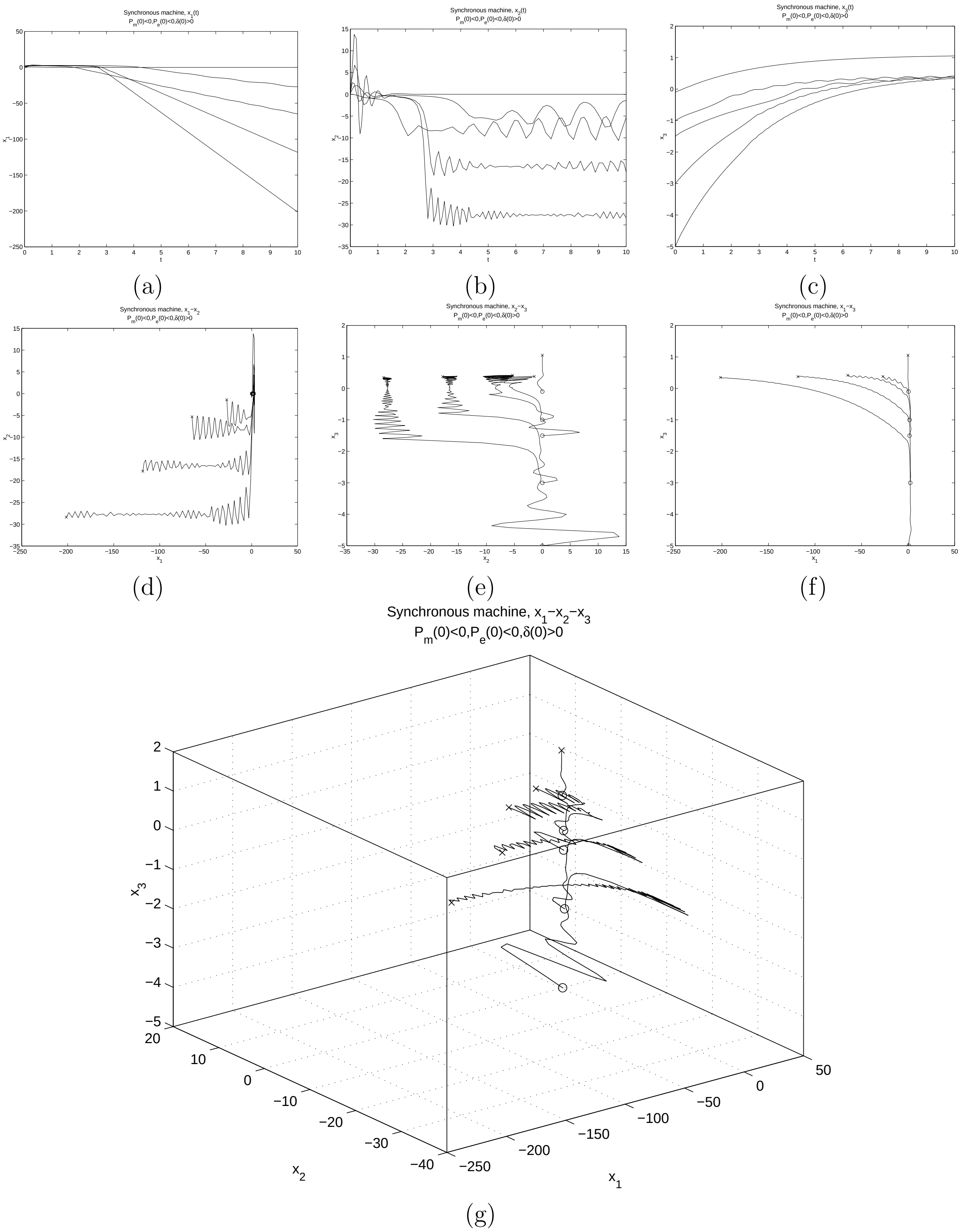


Figure 36: $\delta(0) > 0,\ \dot{\delta}(0) > 0,\ P_e(0) > 0.$

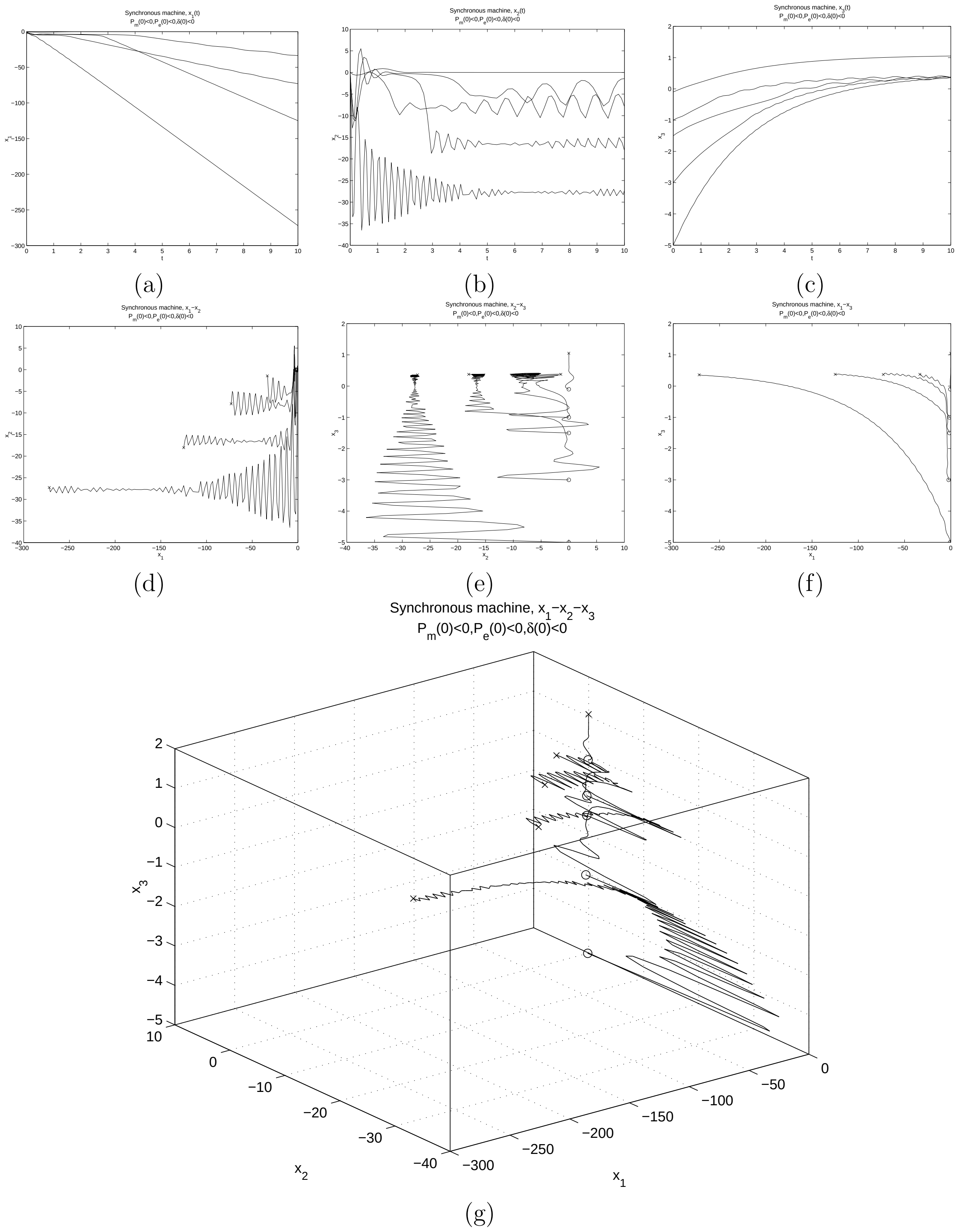


Figure 37: $\delta(0) > 0,\ \dot{\delta}(0) > 0,\ P_e(0) > 0$.

Simulation to study the effect of $\delta(0)$

The next result was simulated with

$$\dot{\delta}(0) = 0, \quad P_e(0) = 2.3,$$

and the following system parameters,

$$P_m = 1.27, \quad H = 0.01, \quad \eta_1 = 2.1, \quad \eta_2 = 2.5, \quad \eta_3 = 1.6, \quad \tau = 7.7, E_{FD} = 1.28, \quad D = 0.052.$$

Figure 38 shows the result when

$$\delta(0) = \frac{n\pi}{6}, \quad \pi = 0, 1, \ldots, 6.$$

Figure 38 shows that the initial value $\delta(0)$ does not have the effect on the steadystate value of $\delta(t)$, further result done on a longer simulation time did show that the steadystate value was

$$\lim_{t\to\infty} \delta(t) \approx \frac{\pi}{5}.$$

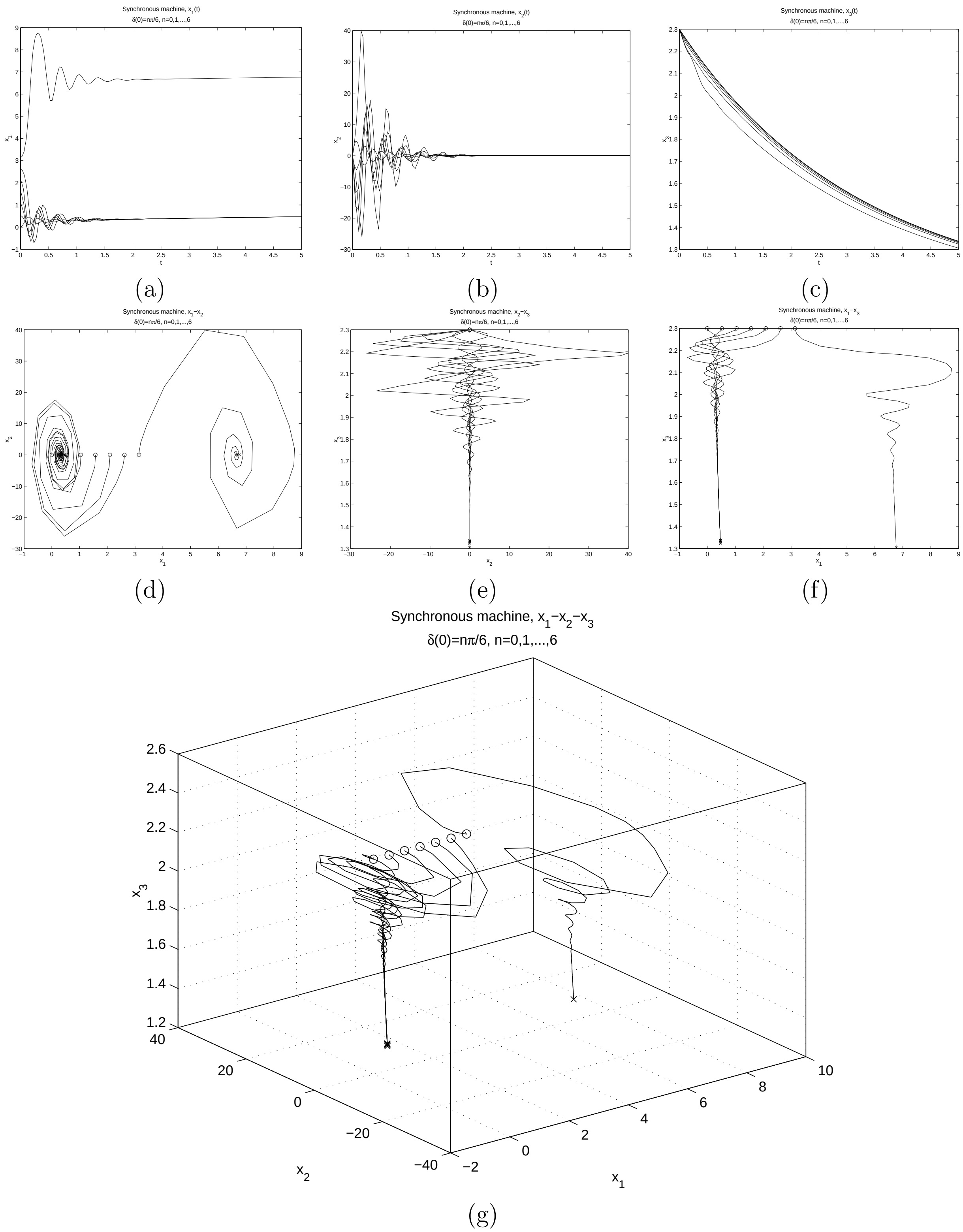


Figure 38: *Synchronous Generator* $P_m = 1.27$, $P_e(0) = 2.3$, $\delta(0) = \frac{n\pi}{6}$, $n = 0, 1, \ldots, 6$.

Simulation to study the effect of P_e

To study the effect of P_e, let

$$P_e = n, \quad n = 0, 1, 2, \ldots, 6.$$

Figure 39 was simulated with

$$\delta(0) = \frac{\pi}{2}, \dot{\delta}(0) = 1,$$

together the following system parameters,

$$P_m = 1.28, \quad H = 0.014, \quad \eta_1 = 1.98, \quad \eta_2 = 2.67, \quad \eta_3 = 1.57, \quad \tau = 9.1, E_{FD} = 1.37, \quad D = 0.0588.$$

Figure 39 shows that

$$\frac{\pi}{4} < \lim_{t\to\infty} \delta(t) \approx 0.75 \text{ rad} < \frac{\pi}{5}.$$

Also,

$$\lim_{t\to\infty} \dot{\delta} = 0$$

except when $P_e(0) = 0$ where $\dot{\delta}(t)$ increased greatly. The steady state values of $P_e(t)$ was

$$\lim_{t\to\infty} P_e \approx 1.$$

Further simulations done with the same system and parameters, but with $P_e(0) < 0$ gave unstable result, ie

$$\lim_{t\to\infty} \delta(t) \to \infty,$$

no matter whether $P_m > 0$ or < 0.

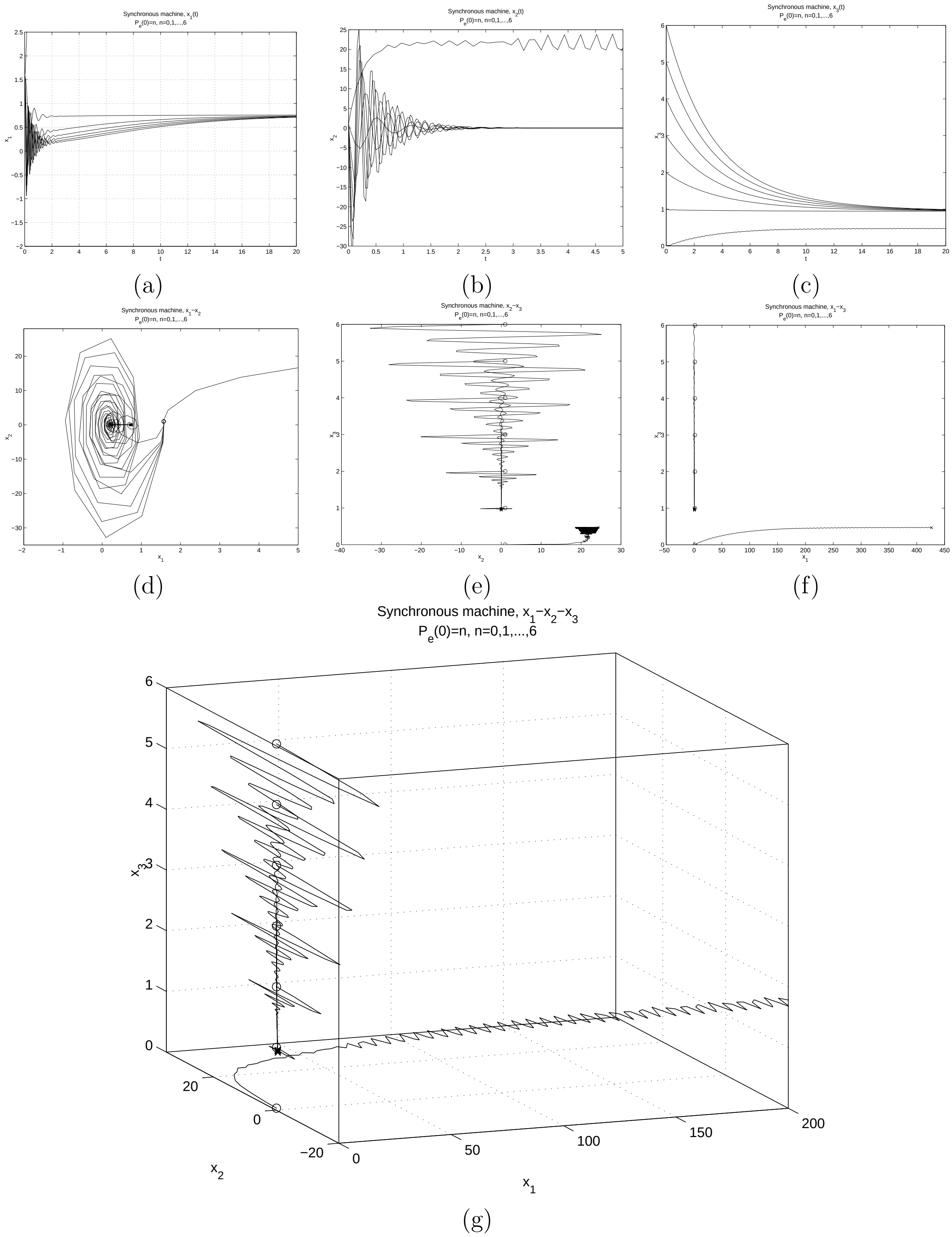


Figure 39: *Synchronous Generator* $P_m = 1.28$, $\delta(0) = 1$, $P_e(0) = n$, $n = 0, 1, \ldots, 6$.

Simulation to study the effect of P_m

Figure 40 shows the result simulated to study the effect of P_m. The system parameters were set to

$$H = 0.021, \quad \eta_1 = 2.98, \quad \eta_2 = 2.67, \quad \eta_3 = 1.74, \quad \tau = 10.1, E_{FD} = 1.12, \quad D = 0.0672,$$

while the initial conditions of the states were set to

$$\delta(0) = \frac{\pi}{4}, \quad \dot{\delta}(0) = 1, \quad P_e(0) = 0.98.$$

Figure 40 shows that there is a maximum point $\max P_m$ for the system to remain stable, and from this simulation

$$\max P_m \approx 2.0.$$

$$P_m > \max P_m$$

will result in

$$\lim_{t\to\infty} \delta(t) \to \infty.$$

The angle $\delta(t)|_{t\to\infty}$ depends on P_m, and $\delta(t)|_{t\to\infty}$ increases with P_m until $\max P_m$ is reached, the corresponding δ at this point (δ_m) from this simulation was

$$\max \delta_m < 0.8 \text{ rad}.$$

Also, P_e at steadystate decreased as P_m increased. When P_m was 2.0 and the system became unstable $\lim_{t\to\infty} P_e$ decreases to below 0.8.

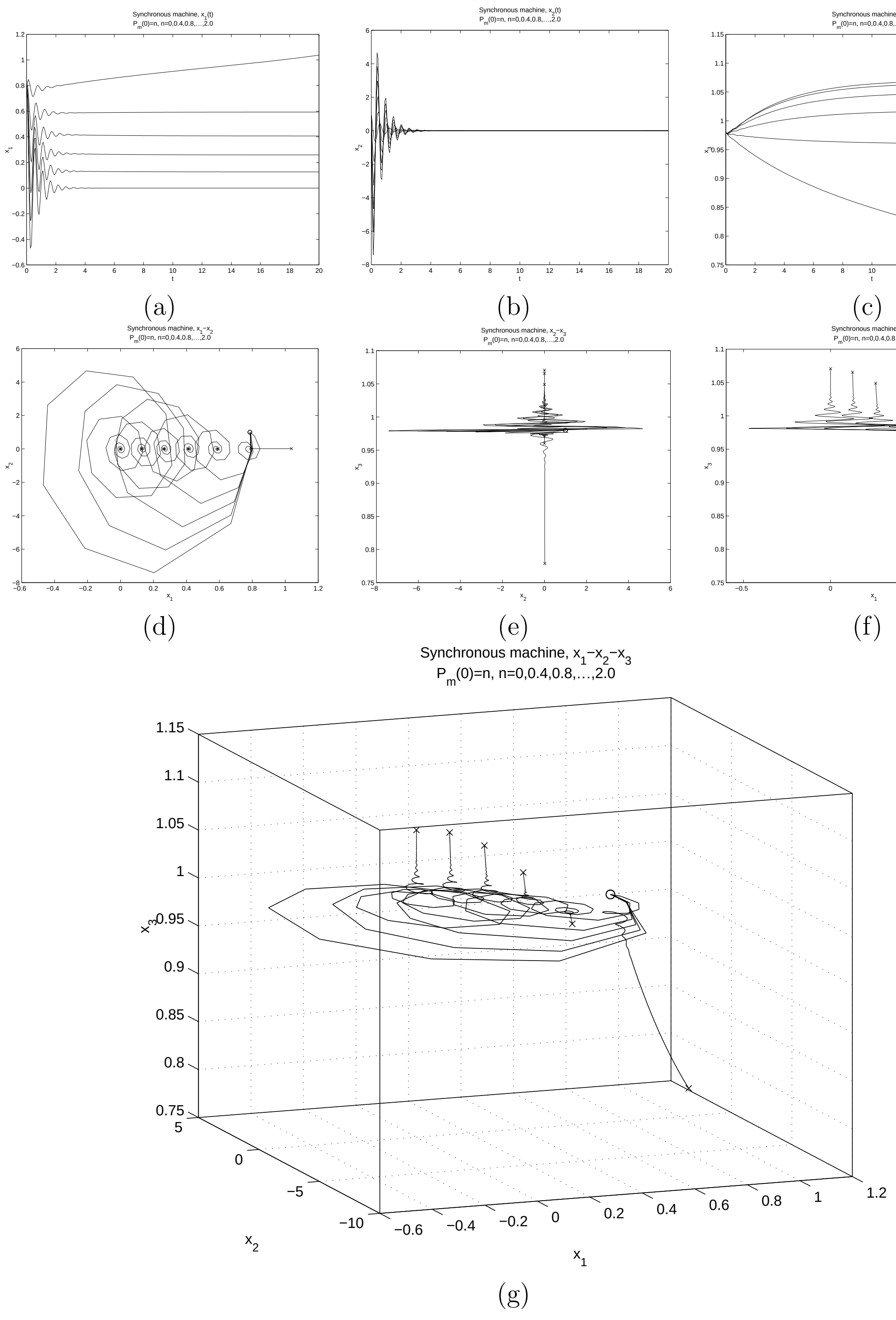


Figure 40: *Synchronous Generator* $P_m = 1.28,\ \delta(0) = 1,\ P_m(0) = n,\ n = 0, 0.4, 0.8 \ldots, 2.0.$

Figure 41 shows a plot between P_m and the steadystate P_e, where I assumed that

$$\lim_{t\to\infty} P_e(t) \approx P_e(200).$$

Figure 41 shows how $P_e|_{t\to\infty}$ decreases sharply at the point $P_m \approx 1.9$. Also,

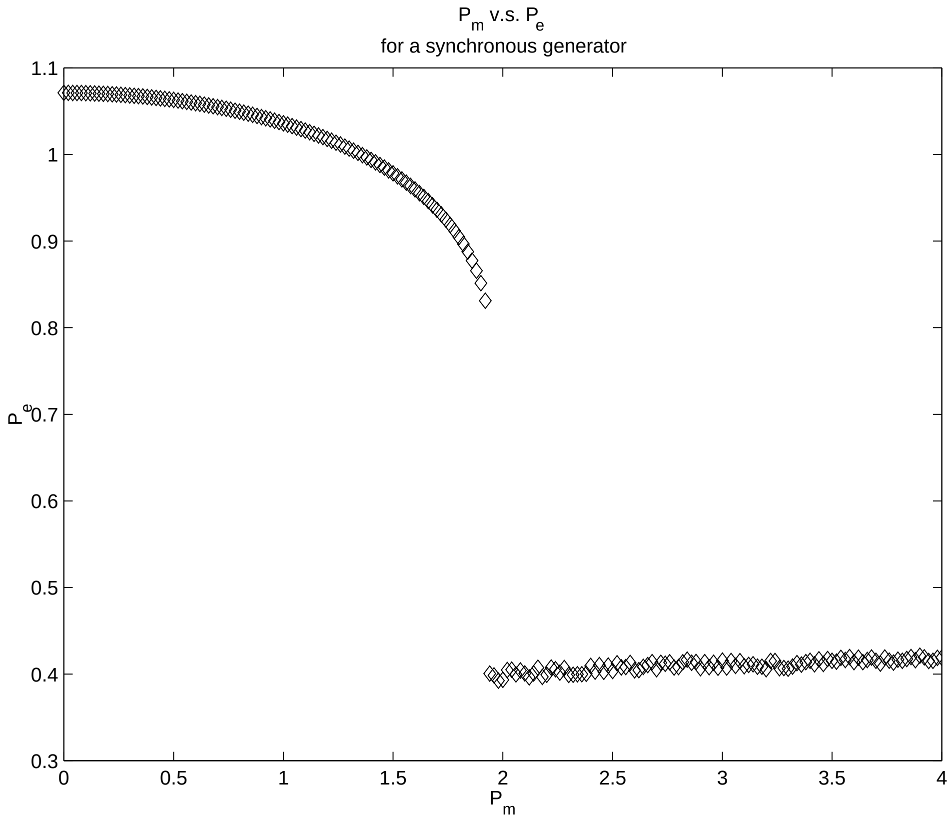
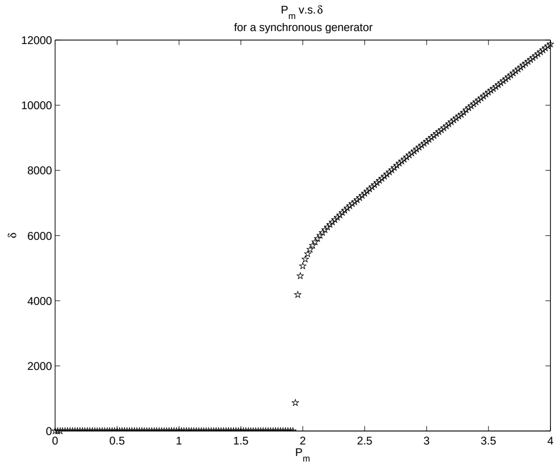


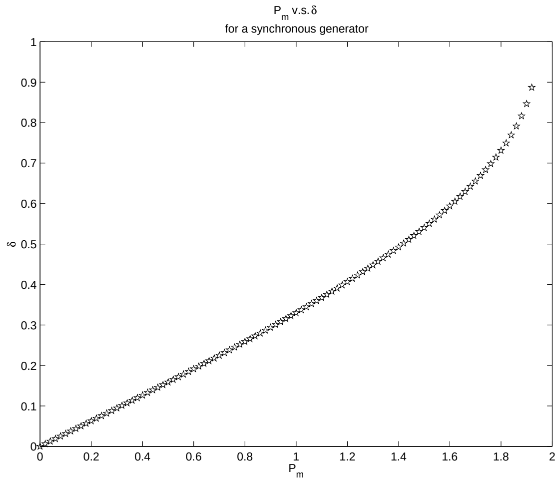
Figure 41: *P_m v.s. $P_e(200)$ of a synchronous generator $P_e(0) = 0.98$, $\delta(0) = 1$, $P_m(0) = n$, $n = 0, 0.03, 0.06, \ldots, 4.0$.*

$$P_e|_{t\to\infty} \approx 0.4, \quad \text{for} \quad P_m \geq 1.9.$$

Figure 42 shows a graph between P_m and δ of the same synchronous generator as above. Figure 42 shows how $\delta(t)$ also runs away



(a)



(b)

Figure 42: P_m *v.s.* δ *synchronous generator* $P_e(0) = 0.98$*,* $\delta(0) = 1$*,* $P_m(0) = n$*,* $n = 0, 0.03, 0.06, \ldots, 4.0$*. (b) is a closed-up detail of (a)*

The saturation of the graph on the right hand side of Figure 42 could be due to the fact that the simulation was done upto $t = 200$ and not $t \to \infty$, according to the approximation mentioned earlier in Equation . (Actually this was confirmed by doing another simulation with a longer simulation time. When $t_{final} = 400$ was used instead of $t_{final} = 200$ the graph of Figure 42 (a) went up to $\delta \approx 2.3 \times 10^4$ instead of $\delta \approx 1.2 \times 10^4$ as in the figure.)

Figure 43 shows a graph between $P_m - P_e$ (which is called the accelerating power of the machine) and δ of the same synchronous generator. Figure 43 shows how the accerating power,

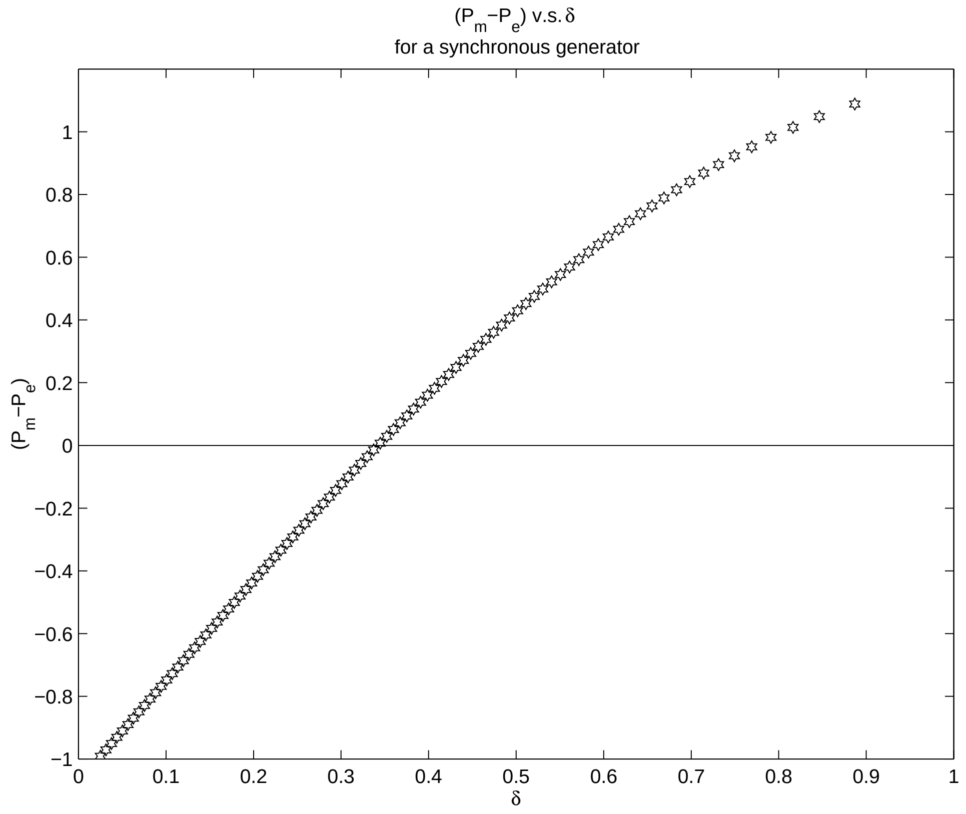


Figure 43: $(P_m - P_e)$ *v.s.* δ *synchronous generator* $P_e(0) = 0.98$, $\delta(0) = 1$, $P_m(0) = n$, $n = 0, 0.03, 0.06, \ldots, 4.0$.

$$P_a = P_m - P_e, \tag{44}$$

swings from negative to positive. From this figure, at

$$\delta \approx 0.7$$

a point is reached where the area between the curve of the graph and the P_a axis adds up to zero. Beyond this point the machine loses synchronism and the speed increases indefinitely. In practice, however, this increase speed may be limitted by physical factors.

Equal area criterion

The equal area criterion [AF77] can give the reason why there exists a maximum $\delta_{\max}$ which $\delta(t)$ may not exceed if the system is to remain stable. The equation is derived in the following manner.

Let P_a be called the accelerating (per unit) power, then

$$
\begin{aligned}
\frac{2H}{\omega_R}\frac{d^2\delta}{dt^2} &= P_m - P_e = \quad P_a \\
\frac{d^2\delta}{dt^2} &= \frac{\omega_R}{2H}P_a.
\end{aligned}
$$

Multiply both sides by $2\frac{d\delta}{dt}$,

$$
\begin{aligned}
2\frac{d^2\delta}{dt^2}\frac{d\delta}{dt} &= \frac{\omega_R}{2H}P_a \cdot 2\frac{d\delta}{dt} \\
\frac{d}{dt}\left[\left(\frac{d\delta}{dt}\right)^2\right] &= \frac{\omega_R}{H}P_a\frac{d\delta}{dt} \\
d\left[\left(\frac{d\delta}{dt}\right)^2\right] &= \frac{\omega_R}{H}P_a d\delta.
\end{aligned}
$$

Integrate both sides,

$$
\begin{aligned}
\left(\frac{d\delta}{dt}\right)^2 &= \frac{\omega_R}{H}\int_{\delta_0}^{\delta} P_a d\delta \\
\frac{d\delta}{dt} &= \sqrt{\frac{\omega_R}{H}\int_{\delta_0}^{\delta} P_a d\delta}. && (45) \\
& && (46)
\end{aligned}
$$

Equation 45 gives the relative speed with respect to a reference frame (which moves at a constant speed.)

For stability,

when rotor is not accelerating, $\frac{d\delta}{dt}$ must be zero, and

when rotor is accelerating, $\frac{d\delta}{dt}$ must oppose rotor motion (relative to the reference frame), and the stability conditions is that

$$
\exists\delta_{\max} \quad \text{s.t.} \quad P_a\left(\delta_{\max}\right) \leq 0, \quad \text{and} \int_{\delta_0}^{\delta} P_a d\delta = 0. \tag{47}
$$

Discussion

Volterra-Lotka ecosystem

Consider a Volterra-Lotka ecosystem model written in state space form as

$$\dot{x_1} = (a - bx_2)\, x_1 \tag{48}$$
$$\dot{x_2} = (cx_1 - d)\, x_2, \tag{49}$$

where

$$a, b, c, d, x_1(t), x_2(t) \in \mathcal{R}^+.$$

Some knowledge about this system

The equilibrium point can be found by letting

$$\dot{x} = f(x) = 0,$$

which leads to

$$x_1 = \frac{d}{c}, \quad x_2 = \frac{a}{b}$$

as the equilibrium point.

The nature of this equilibrium point can be found by looking at the eigenvalues of the matrix A of the system linearized around the equilibrium point itself, that is

$$\dot{x} = Ax + Bu, \tag{50}$$

where A is evaluated from the Jacobian matrix at the equilibrium point, ie.

$$A = \left.\frac{\partial f}{\partial x}\right|_{x=\left(\frac{d}{c},\frac{a}{b}\right)} = \left.\begin{bmatrix} a - bx_2 & -bx_1 \\ cx_2 & cx_1 - d \end{bmatrix}\right|_{x=\left(\frac{d}{c},\frac{a}{b}\right)} = \begin{bmatrix} 0 & -\frac{bd}{c} \\ \frac{ac}{b} & 0 \end{bmatrix}. \tag{51}$$

The eigenvalues of A are then

$$\begin{bmatrix} \sqrt{-ad} \\ -\sqrt{-ad} \end{bmatrix},$$

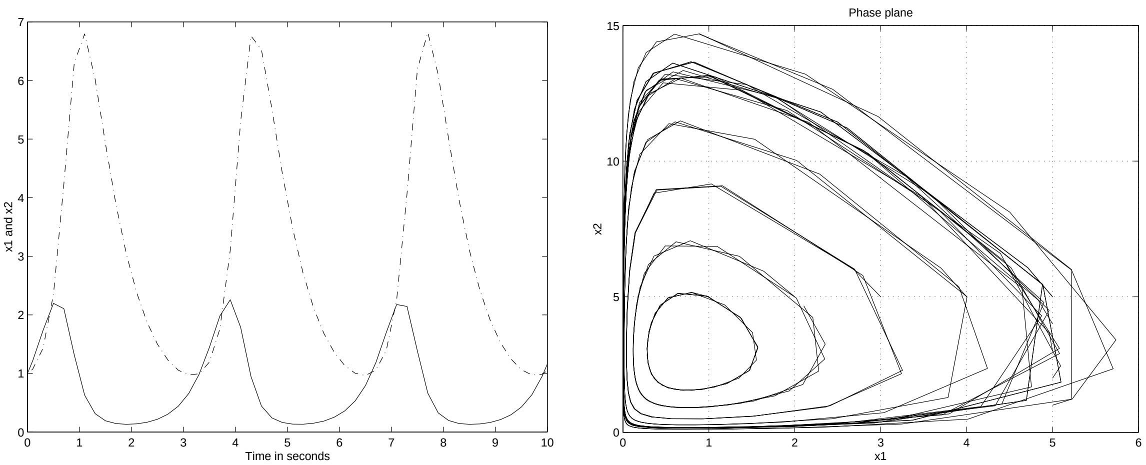
which when $a, d > 0$ leads to the eigenvalues at

$$\jmath\sqrt{ad}.$$

The eigenvalues are pure imaginaries means that the equilibrium point is a *center* of trajectories.

Figure 44 shows the result of simulations done on this system having

$$a = 3, \quad b = 1, \quad c = 2, \quad d = 1.5.$$



(a) (b)

Figure 44: *A Volterra-Lotka ecosystem model, x_2 is shown with a dotted line ($a = 3$, $b = 1$, $c = 2$, $d = 1.5$), (a) state variables, (b) state plane.*

With a signum function feedback

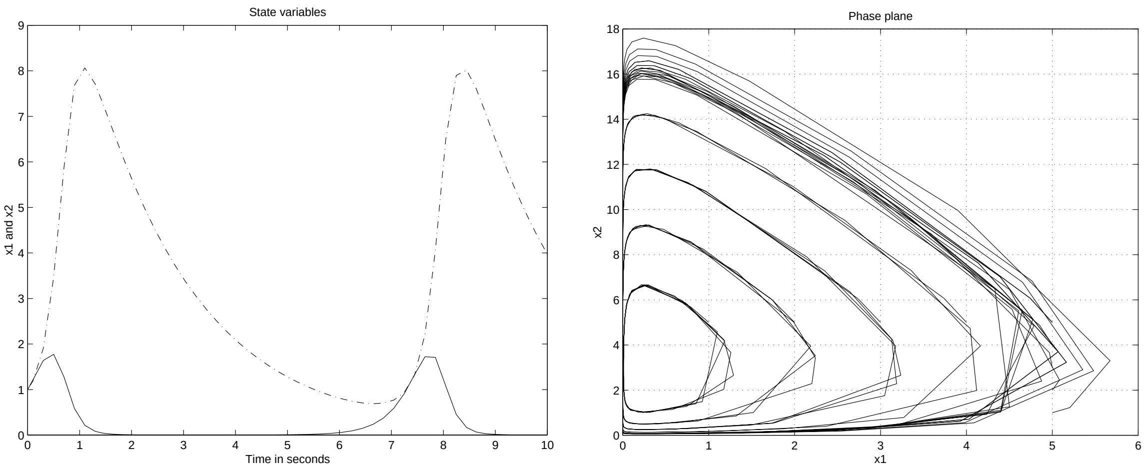
If an input

$$u = k\,\mathrm{sgn}\,(\mathrm{x}_1 + \mathrm{x}_2)$$

was applied to this system in the following manner

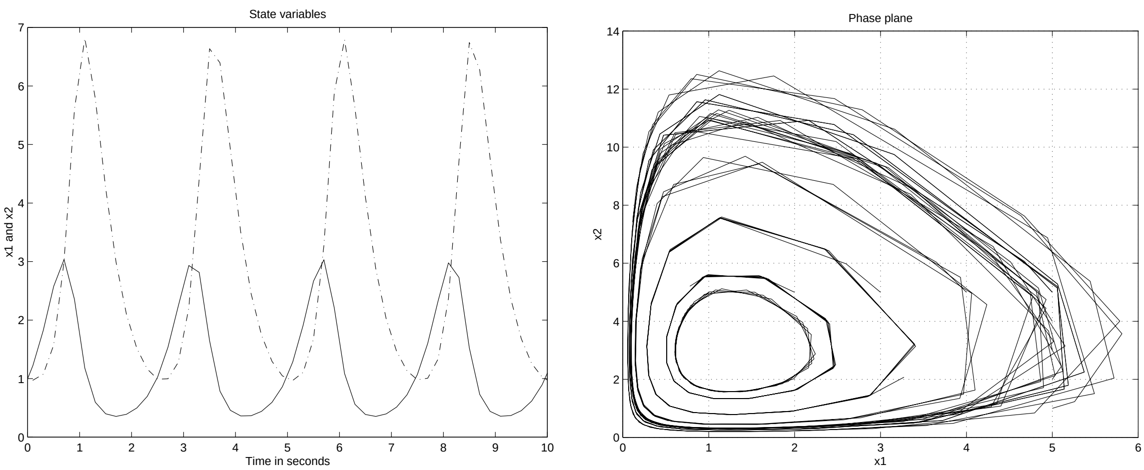
$$\begin{aligned} \dot{x_1} &= (a - bx_2)\,x_1 && (52)\\ \dot{x_2} &= (cx_1 - d)\,x_2 + u && (53) \end{aligned}$$

the response was that shown in Figure 45 ($k = 1$), and Figure 46 ($k = -1$).



(a) (b)

Figure 45: *A Volterra-Lotka ecosystem model with* $u = \mathrm{sgn}\,(\mathrm{x}_1 + \mathrm{x}_2)$*,* x_2 *is shown with dotted line (*$a = 3$*,* $b = 1$*,* $c = 2$*,* $d = 1.5$*), (a) shows state variables, (b) state plane*



(a) (b)

Figure 46: *A Volterra-Lotka ecosystem model with* $u = -\mathrm{sgn}\,(\mathrm{x}_1 + \mathrm{x}_2)$*,* x_2 *is shown with dotted line (*$a = 3$*,* $b = 1$*,* $c = 2$*,* $d = 1.5$*), (a) shows state variables, (b) state plane*

Two signum function inputs

Incorporating into Equaiton 48 and 49 the constraints (from reality) that

$$x_1 \geq 0, \quad x_2 \geq 0, \tag{54}$$

and also add inputs

$$\begin{aligned} u_1 &= \kappa_1 \mathrm{sgn}\left(\vartheta_1 x_1 + x_2 - \varrho_1\right) & (55) \\ u_1 &= \kappa_2 \mathrm{sgn}\left(\vartheta_2 x_1 + x_2 - \varrho_2\right) & (56) \end{aligned}$$

to obtain

$$\begin{aligned} \dot{x_1} &= \left(a - b\max\left(x_2, 0\right)\right)\max\left(x_1, 0\right) + \kappa_1 \mathrm{sgn}\left(\vartheta_1 x_1 + x_2\right) & (57) \\ \dot{x_2} &= \left(c\max\left(x_1, 0\right) - d\right)\max\left(x_2, 0\right) + \kappa_2 \mathrm{sgn}\left(\vartheta_2 x_1 + x_2\right). & (58) \end{aligned}$$

Here κ_1, κ_2, ϑ_1, ϑ_2, ϱ_1, and ϱ_2 are constants and

$$\vartheta_1, \vartheta_2, \varrho_1, \varrho_2 \geq 0.$$

The following simulation used the set of parameters

$$a = 2.4, b = 1.45, c = 7.3, d = 2.6, \vartheta_1 = 0.3, \vartheta_2 = 21, \varrho_1 = 6.7, \varrho_2 = 57.$$

For the feedback gain κ_1 and κ_2, four values were used, they were the combinations of

$$\kappa_1 = \pm 5.6, \quad \text{and} \kappa_2 = \pm 8.3.$$

The result from the simulations is shown in Figure 47, 48, 49, and 50. The result from simulation when only either one of the two κ's was applied was also obtained, and is shown in Figure 51, 52, 53, and 54. In all cases, the initial conditions were taken as

$$(x_1(0), x_2(0)) = (0.1, 6.9), (2.1, 4.9), (4.1, 2.9), (6.1, 0.9).$$

The plots of $u_1(t)$ and $u_2(t)$ are shown in this order of initial conditions, ie. left $\rightarrow$ right and then top $\rightarrow$ bottom.

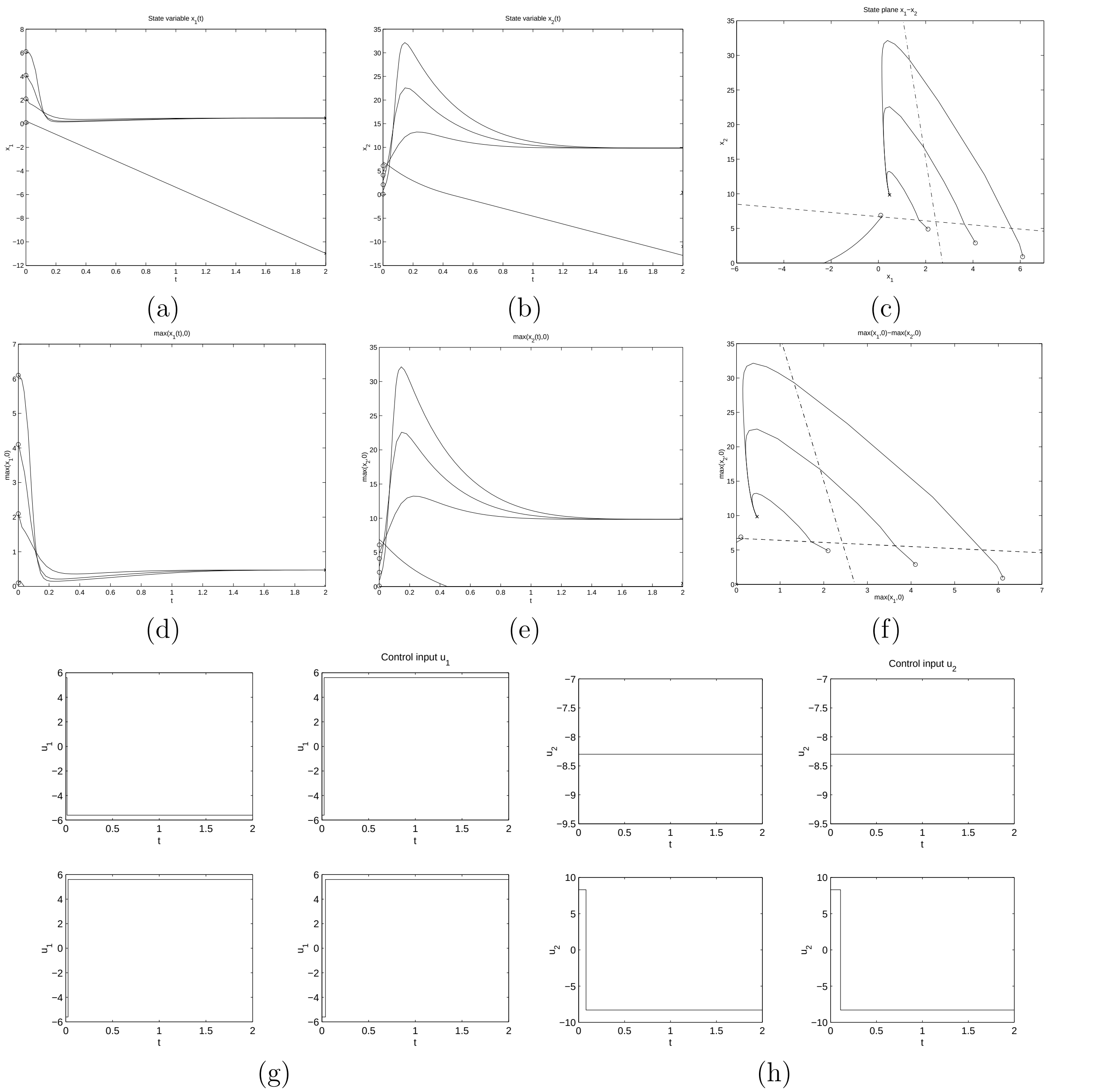


Figure 47: *Two signum inputs Volterra-Lotka system* $\kappa_1 = 5.6, \kappa_2 = 8.3$ *(a)* $x_1(t)$*, (b)* $x_2(t)$*, (c)* x_1 *v.s.* x_2*, (d)* $\max(x_1, 0)(t)$*, (e)* $\max(x_2, 0)(t)$*, (f)* $\max(x_1, 0)(t)$ *v.s.* $\max(x_2, 0)(t)$*, (g)* $u_1(t)$*, (h)* $u_2(t)$

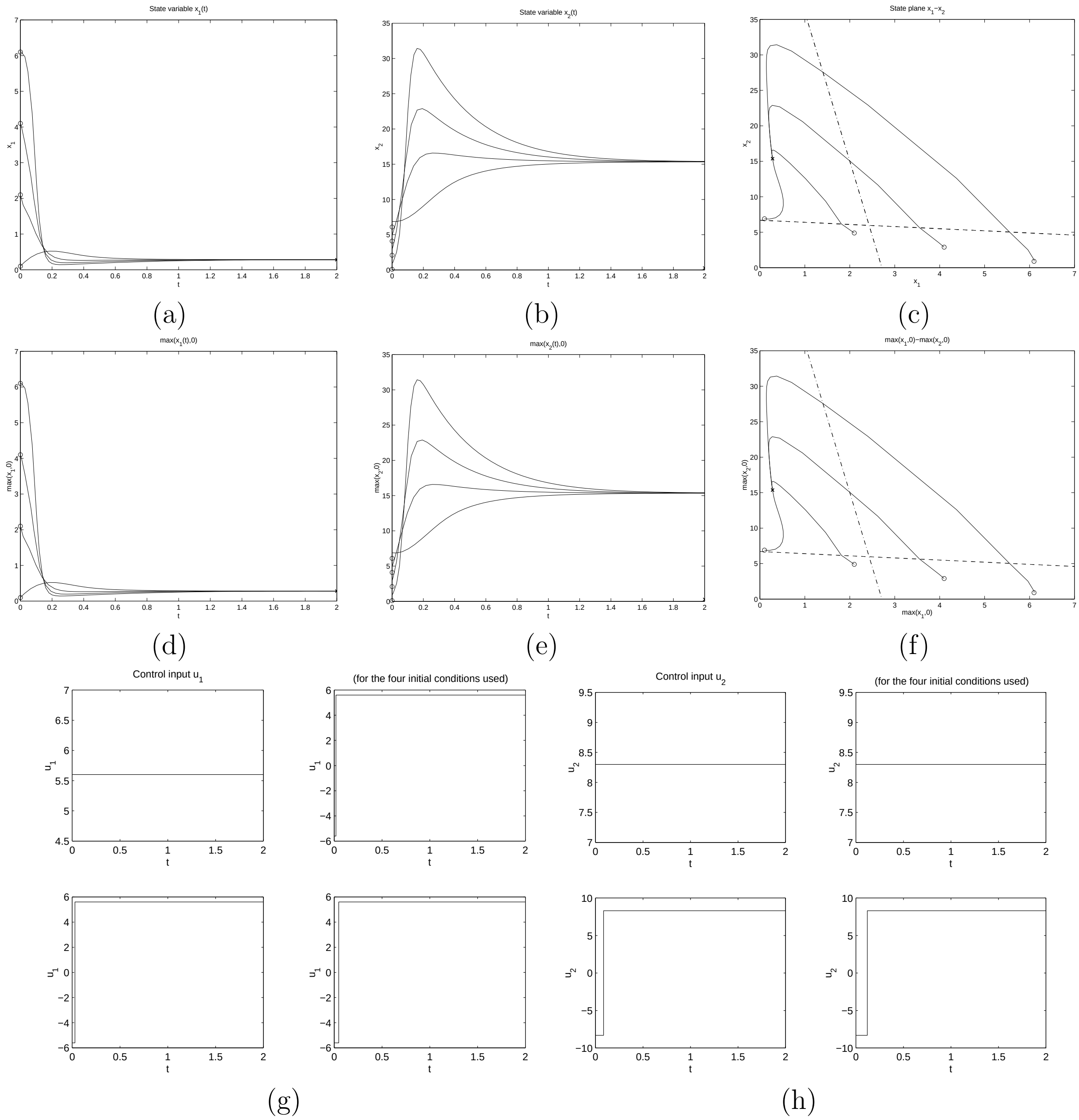


Figure 48: *Two signum inputs Volterra-Lotka system* $\kappa_1 = 5.6, \kappa_2 = -8.3$ *(a)* $x_1(t)$, *(b)* $x_2(t)$, *(c)* x_1 *v.s.* x_2, *(d)* $\max(x_1, 0)(t)$, *(e)* $\max(x_2, 0)(t)$, *(f)* $\max(x_1, 0)(t)$ *v.s.* $\max(x_2, 0)(t)$, *(g)* $u_1(t)$, *(h)* $u_2(t)$

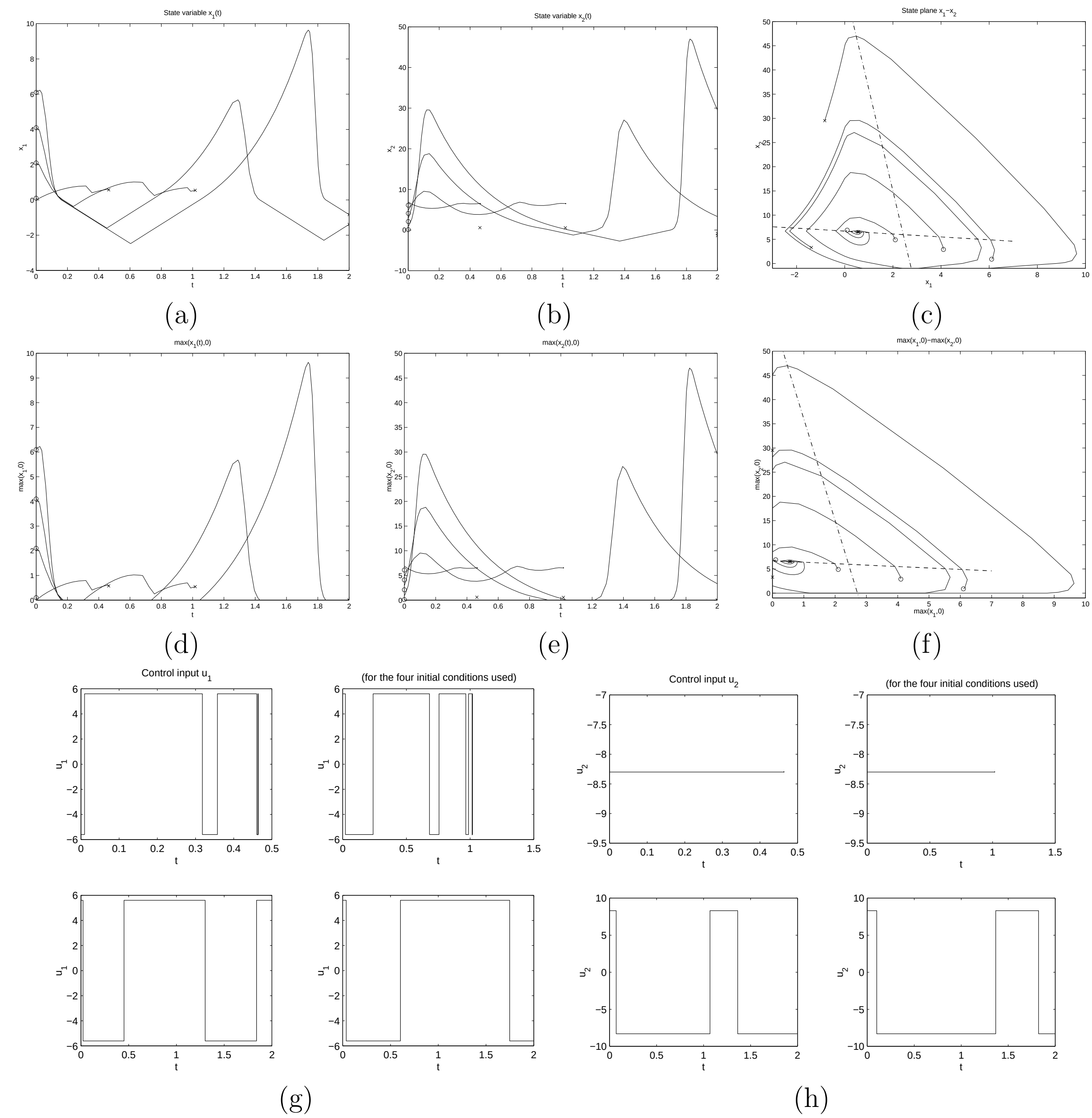


Figure 49: *Two signum inputs Volterra-Lotka system* $\kappa_1 = -5.6, \kappa_2 = 8.3$ *(a)* $x_1(t)$, *(b)* $x_2(t)$, *(c)* x_1 *v.s.* x_2, *(d)* $\max(x_1, 0)(t)$, *(e)* $\max(x_2, 0)(t)$, *(f)* $\max(x_1, 0)(t)$ *v.s.* $\max(x_2, 0)(t)$, *(g)* $u_1(t)$, *(h)* $u_2(t)$

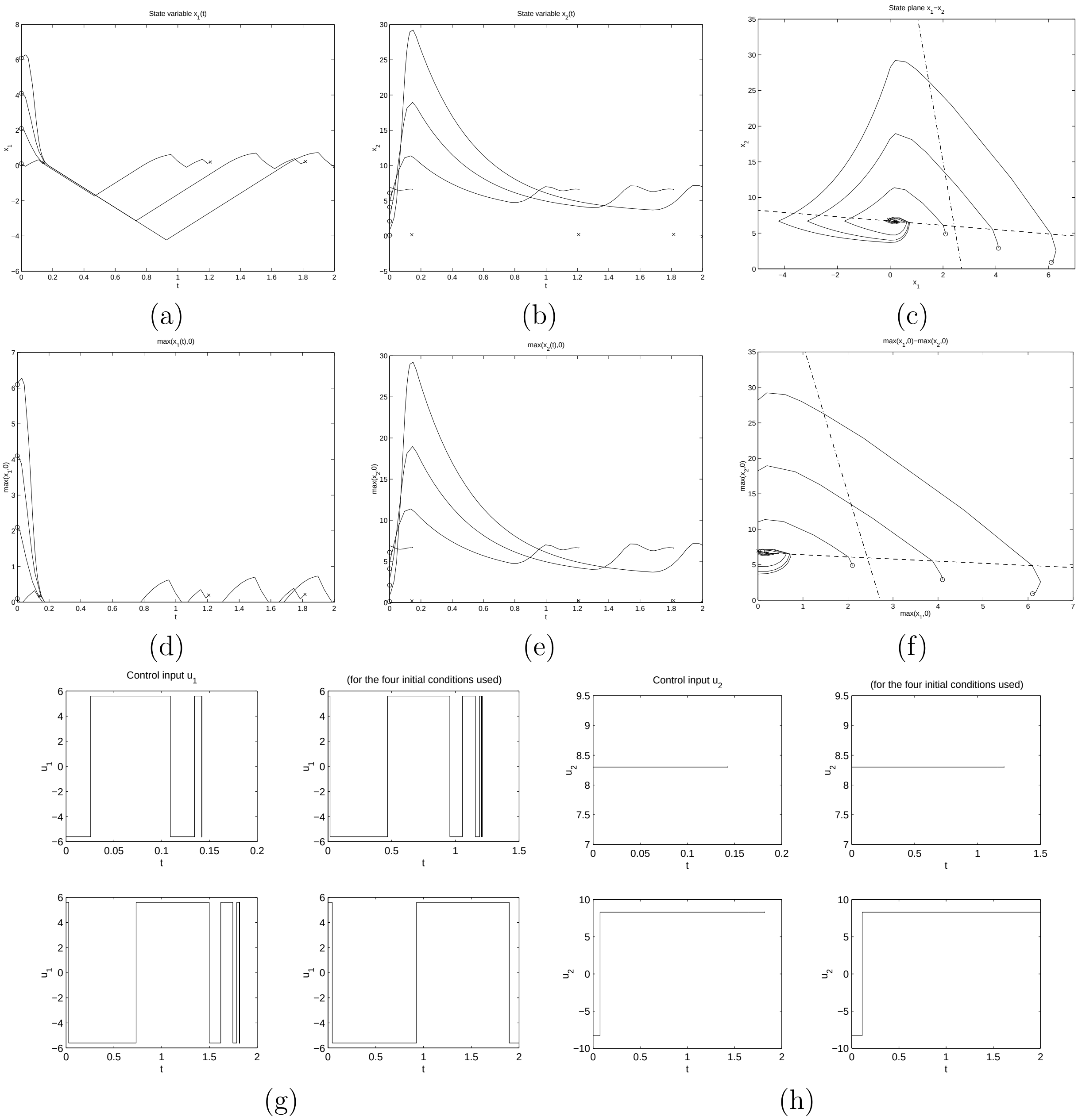


Figure 50: *Two signum inputs Volterra-Lotka system* $\kappa_1 = -5.6, \kappa_2 = -8.3$ *(a)* $x_1(t)$*, (b)* $x_2(t)$*, (c)* x_1 *v.s.* x_2*, (d)* $\max(x_1, 0)(t)$*, (e)* $\max(x_2, 0)(t)$*, (f)* $\max(x_1, 0)(t)$ *v.s.* $\max(x_2, 0)(t)$*, (g)* $u_1(t)$*, (h)* $u_2(t)$

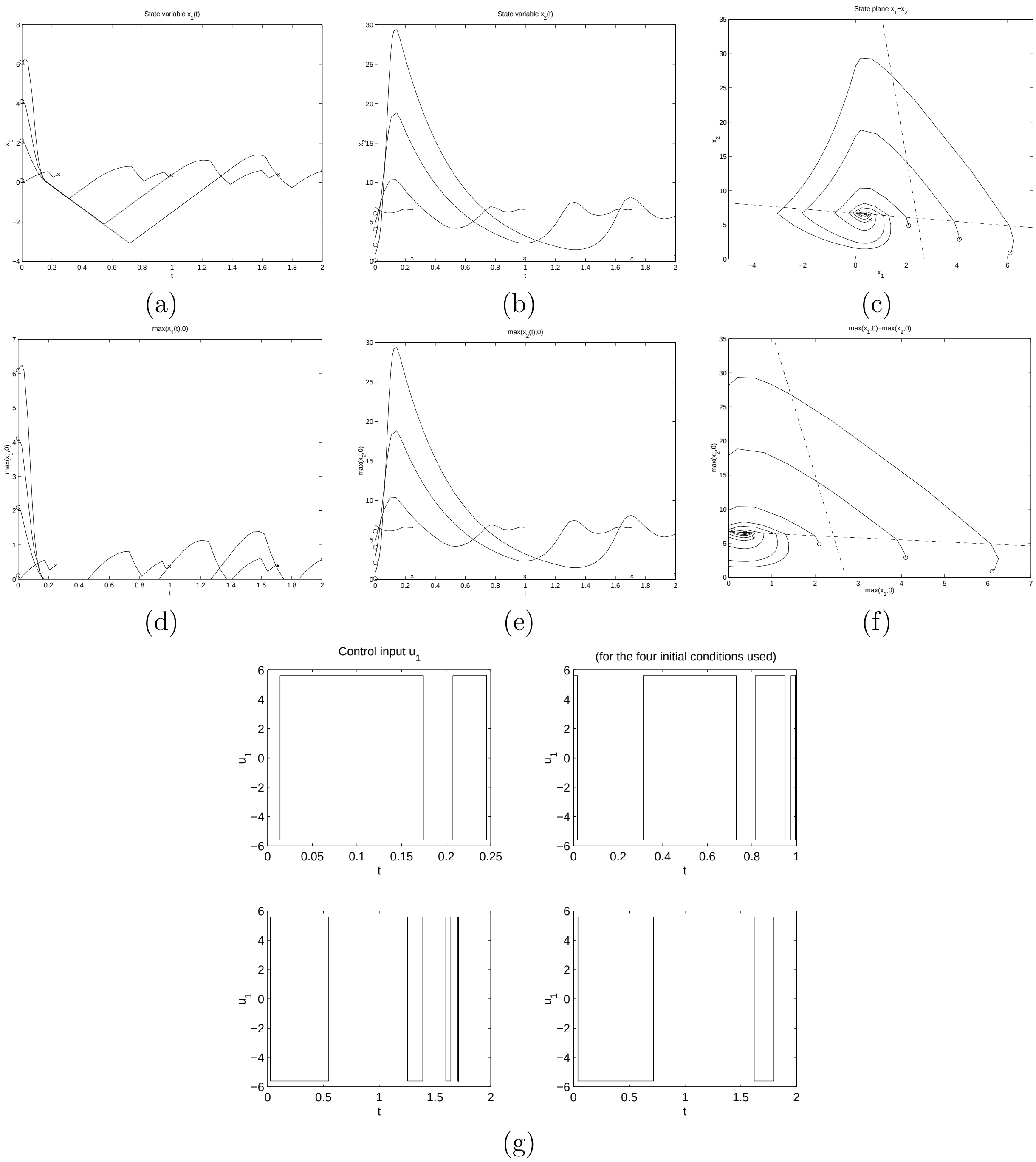


Figure 51: *Two signum inputs Volterra-Lotka system* $\kappa_1 = -5.6, \kappa_2 = 0$ *(a)* $x_1(t)$*, (b)* $x_2(t)$*, (c)* x_1 *v.s.* x_2*, (d)* $\max(x_1, 0)(t)$*, (e)* $\max(x_2, 0)(t)$*, (f)* $\max(x_1, 0)(t)$ *v.s.* $\max(x_2, 0)(t)$*, (g)* $u_1(t)$

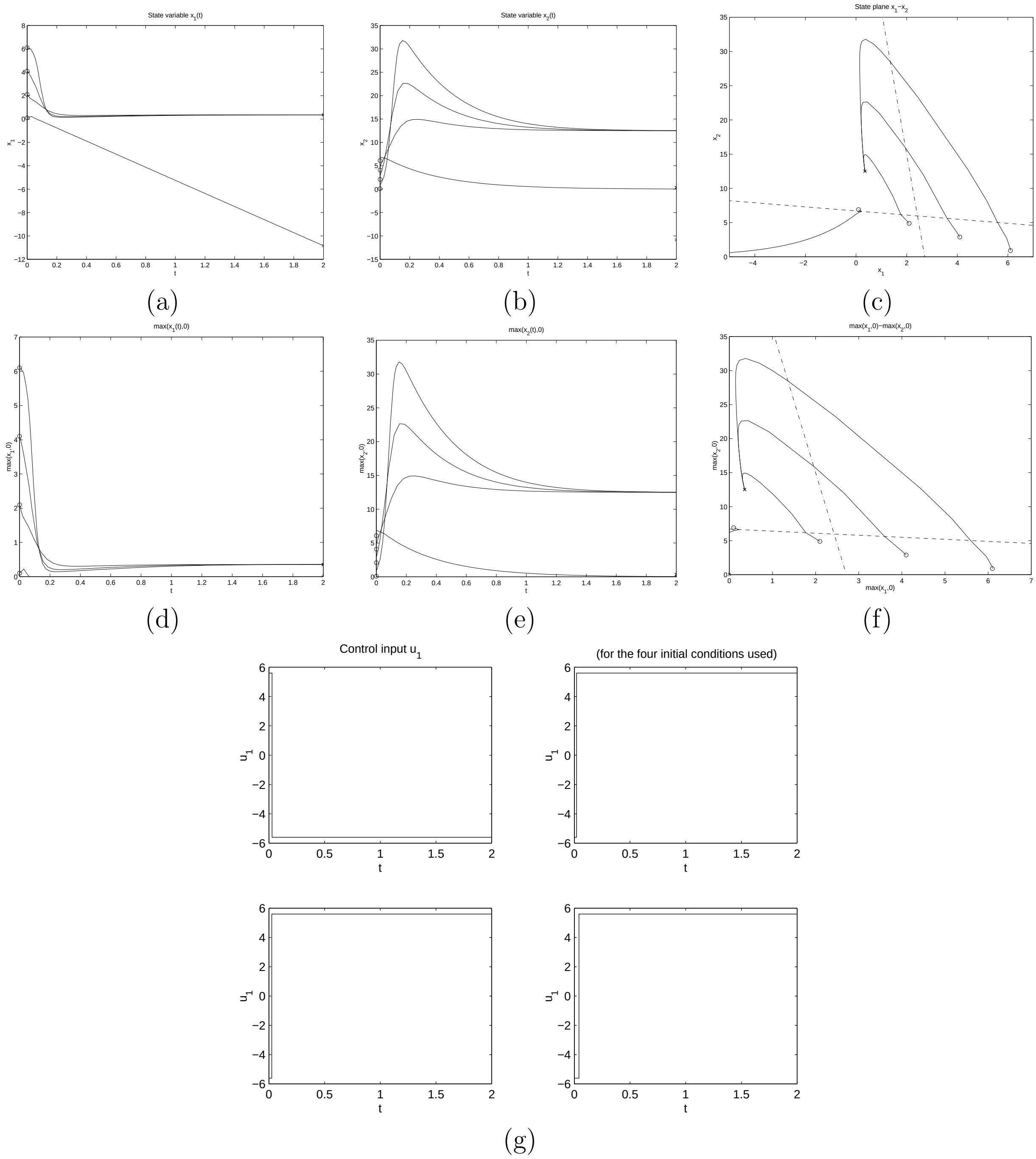


Figure 52: *Two signum inputs Volterra-Lotka system* $\kappa_1 = 5.6, \kappa_2 = 0$ *(a)* $x_1(t)$*, (b)* $x_2(t)$*, (c)* x_1 *v.s.* x_2*, (d)* $\max(x_1, 0)(t)$*, (e)* $\max(x_2, 0)(t)$*, (f)* $\max(x_1, 0)(t)$ *v.s.* $\max(x_2, 0)(t)$*, (g)* $u_1(t)$

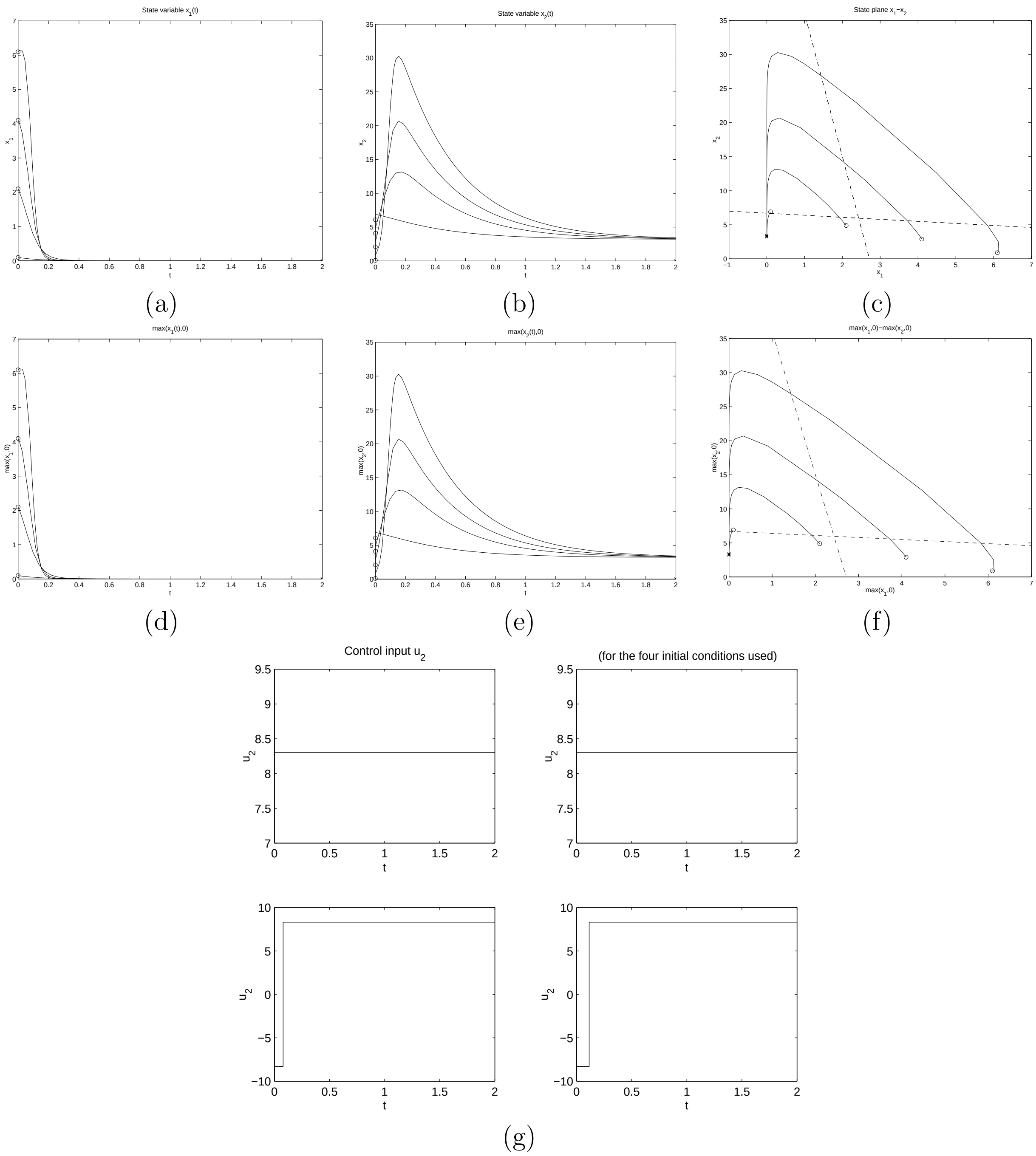


Figure 53: *Two signum inputs Volterra-Lotka system* $\kappa_1 = 0, \kappa_2 = -8.3$ *(a)* $x_1(t)$*, (b)* $x_2(t)$*, (c)* x_1 *v.s.* x_2*, (d)* $\max(x_1, 0)(t)$*, (e)* $\max(x_2, 0)(t)$*, (f)* $\max(x_1, 0)(t)$ *v.s.* $\max(x_2, 0)(t)$*, (g)* $u_2(t)$

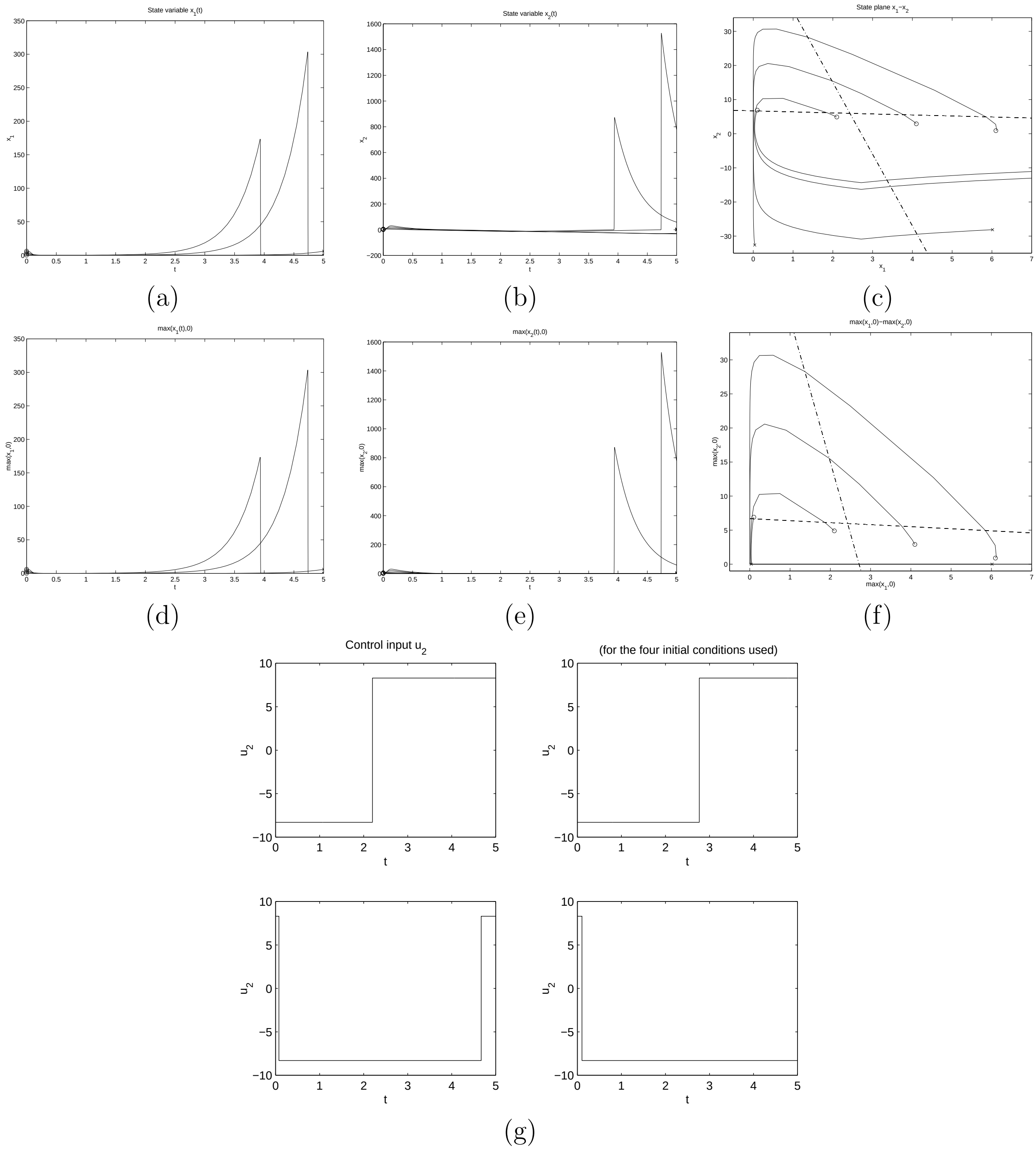


Figure 54: *Two signum inputs Volterra-Lotka system* $\kappa_1 = 0, \kappa_2 = 8.3$ *(a)* $x_1(t)$*, (b)* $x_2(t)$*, (c)* x_1 *v.s.* x_2*, (d)* $\max(x_1, 0)(t)$*, (e)* $\max(x_2, 0)(t)$*, (f)* $\max(x_1, 0)(t)$ *v.s.* $\max(x_2, 0)(t)$*, (g)* $u_2(t)$

Sensitivity analysis

To do a sensitivity analysis let (see also [Kha96])

$$\dot{x} = f(t, x, \lambda), \quad x(t_0) = x_0.$$

The state variables can be written as

$$x(t, \lambda) = x_0 + \int_{t_0}^{t} f(s, x(s, \lambda), \lambda) ds.$$

Take partial derivative with respect to λ,

$$x_\lambda(t, \lambda) = \int_{t_0}^{t} \left[\frac{\partial f}{\partial x}(s, x(s, \lambda), \lambda) x_\lambda(s, \lambda) + \frac{\partial f}{\partial \lambda}(s, x(s, \lambda), \lambda) \right] ds, \tag{59}$$

where

$$x_\lambda(t, \lambda) = \frac{\partial x(t, \lambda)}{\partial \lambda}, \quad \text{and} \frac{\partial x_0}{\partial \lambda} = 0.$$

Differentiate with respect to t,

$$\frac{\partial}{\partial t} x_\lambda(t, \lambda) = A(t, \lambda) x_\lambda(t, \lambda) + B(t, \lambda), \quad x_\lambda(t_0, \lambda) = 0, \tag{60}$$

where

$$A(t, \lambda) x_\lambda(t, \lambda) = \left. \frac{\partial f(t, x, \lambda)}{\partial x} \right| x = x(t, \lambda), \quad B(t, \lambda) = \left. \frac{\partial f(t, x, \lambda)}{\partial \lambda} \right| x = x(t, \lambda).$$

Let the sensitivity function be $S(t) = x_\lambda(t, \lambda_0)$ and obtain the sensitivity equation

$$\dot{S}(t) = A(t, \lambda_0) S(t) + B(t, \lambda_0), \quad S(t_0) = 0. \tag{61}$$

From Equation 48 and 49 the Jacobian matrix with respect to the state variables is

$$\frac{\partial f}{\partial x} = \begin{bmatrix} a - bx_2 & -bx_1 \\ cx_2 & -d + cx_1 \end{bmatrix}$$

and the Jacobian matrix with respect to the parameters is

$$\frac{\partial f}{\partial \lambda} = \begin{bmatrix} x_1 & -x_1 x_2 & 0 & 0 \\ 0 & 0 & x_1 x_2 & x_2 \end{bmatrix}.$$

Let the nominal values of parameters are

$$a = b = c = d = 1,$$

to obtain

$$\left. \frac{\partial f}{\partial x} \right|_{\text{nominal}} = \begin{bmatrix} 1 - x_2 & -x_1 \\ x_2 & -1 + x_1 \end{bmatrix}, \quad \left. \frac{\partial f}{\partial \lambda} \right|_{nominal} = \begin{bmatrix} x_1 & -x_1 x_2 & 0 & 0 \\ 0 & 0 & x_1 x_2 & x_2 \end{bmatrix}.$$

Let the sensitivity function be

$$S = \begin{bmatrix} x_3 & x_5 & x_7 & x_9 \\ x_4 & x_6 & x_8 & x_{10} \end{bmatrix} = \left. \begin{bmatrix} \frac{\partial x_1}{\partial a} & \frac{\partial x_1}{\partial b} & \frac{\partial x_1}{\partial c} & \frac{\partial x_1}{\partial d} \\ \frac{\partial x_2}{\partial a} & \frac{\partial x_2}{\partial b} & \frac{\partial x_2}{\partial c} & \frac{\partial x_2}{\partial d} \end{bmatrix} \right|_{\text{nominal}}.$$

Then for the purpose of sensitivity analysis, one simulates the system

$$
\begin{array}{rclrcl}
\dot{x_1} & = & (1-x_2)\,x_1 & x_1(0) & = & x_{10} \\
\dot{x_2} & = & (x_1-1)\,x_2 & x_2(0) & = & x_{20} \\
\dot{x_3} & = & x_3 - x_2x_3 - x_1x_4 + x_1 & x_3(0) & = & x_{30} \\
\dot{x_4} & = & x_2x_3 - x_4 + x_1x_4 & x_4(0) & = & x_{40} \\
\dot{x_5} & = & x_5 - x_2x_5 - x_1x_6 - x_1x_2 & x_5(0) & = & x_{50} \\
\dot{x_6} & = & x_2x_5 - x_6 + x_1x_6 & x_6(0) & = & x_{60} \\
\dot{x_7} & = & x_7 - x_2x_7 - x_1x_8 & x_7(0) & = & x_{70} \\
\dot{x_8} & = & x_2x_7 - x_8 + x_1x_8 + x_1x_2 & x_8(0) & = & x_{80} \\
\dot{x_9} & = & x_9 - x_2x_9 - x_1x_{10} & x_9(0) & = & x_{90} \\
\dot{x_{10}} & = & x_2x_9 - x_{10} + x_1x_{10} - x_2 & x_{10}(0) & = & x_{100}
\end{array}
$$

Synchronous machine connected to an infinite bus

When a synchronous machine is connected to an infinite bus[1] via a transmission line the current model is (see also [Tiy98])

$$
\left[\begin{array}{c} \dot{i}_d \\ \dot{i}_F \\ \dot{i}_D \\ \dot{i}_q \\ \dot{i}_Q \\ \hline \dot{\omega} \\ \dot{\delta} \end{array}\right]
=
\left[\begin{array}{ccccc|cc} \multicolumn{5}{c|}{-\hat{L}^{-1}\left(\hat{R}+\omega\hat{N}\right)} & \multicolumn{2}{c}{0} \\ \hline -\frac{L_d i_q}{3\tau_j} & -\frac{kM_F i_q}{3\tau_j} & -\frac{kM_D i_q}{3\tau_j} & \frac{L_d i_q}{3\tau_j} & \frac{kM_Q i_d}{3\tau_j} & -\frac{D}{3\tau_j} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right]
\left[\begin{array}{c} i_d \\ i_F \\ i_D \\ i_q \\ i_Q \\ \hline \omega \\ \delta \end{array}\right] +
$$

$$
\left[\begin{array}{c|cc} -\hat{L}^{-1} & \multicolumn{2}{c}{0} \\ \hline 0 & 1 & 0 \\ & 0 & 1 \end{array}\right]
\left[\begin{array}{c} -K\sin\gamma \\ -v_F \\ 0 \\ \hline K\cos\gamma \\ 0 \\ \hline \frac{T_m}{\tau_j} \\ -1 \end{array}\right], \tag{62}
$$

where r, L_d, and L_q in L, R, and N are replaced by

$$
\hat{R} = r + R_e, \quad \hat{L}_d = L_d + L_e, \quad \hat{L}_q = L_q + L_e, \tag{63}
$$

R_e and L_e being resistance and inductance of the transmission line respectively, and

$$
\gamma = \delta - \alpha. \tag{64}
$$

The flux linkage model is

$$
T\dot{x} = Cx + D, \tag{65}
$$

where

$$
x = \left[\begin{array}{ccccccc} \lambda_d & \lambda_F & \lambda_D & \lambda_q & \lambda_Q & \omega & \delta \end{array}\right]^T, \tag{66}
$$

$$
T = \left[\begin{array}{ccc|cc|cc} 1+\frac{L_e}{l_d}\left(1-\frac{L_{MD}}{l_d}\right) & -\frac{L_e L_{MD}}{l_d l_F} & -\frac{l_e L_{MD}}{l_d l_F} & & & & \\ 0 & 1 & 0 & \multicolumn{2}{c|}{0} & \multicolumn{2}{c}{0} \\ 0 & 0 & 1 & & & & \\ \hline & & & 1+\frac{L_e}{l_q}\left(1-\frac{L_{MQ}}{l_q}\right) & -\frac{L_e L_{MQ}}{l_q l_Q} & & \\ \multicolumn{3}{c|}{0} & 0 & 1 & \multicolumn{2}{c}{0} \\ \hline \multicolumn{3}{c|}{0} & \multicolumn{2}{c|}{0} & 1 & 0 \\ & & & & & 0 & 1 \end{array}\right], \tag{67}
$$

[1] An infinite bus is a constant-voltage, constant-frequency bus (source or sink)

$C =$

$$
\left[\begin{array}{ccc|cc|cc}
-\frac{\hat{R}}{l_d}\left(1-\frac{L_{MD}}{l_d}\right) & \frac{\hat{R}L_{MD}}{l_d l_F} & \frac{\hat{R}L_{MD}}{l_d l_D} & -\omega\left[1+\frac{L_e}{l_q}\left(1-\frac{L_{MQ}}{l_q}\right)\right] & \frac{\omega L_e L_{MQ}}{l_q l_Q} & 0 & 0 \\
\frac{r_F L_{MD}}{l_F l_d} & -\frac{r_F}{l_F}\left(1-\frac{L_{MD}}{l_F}\right) & \frac{r_F L_{MD}}{l_F l_D} & 0 & 0 & 0 & 0 \\
\frac{r_D L_{MD}}{l_D l_d} & \frac{r_D L_{MD}}{l_D l_F} & -\frac{r_D}{l_D}\left(1-\frac{L_{MD}}{l_D}\right) & 0 & 0 & 0 & 0 \\
\hline
\omega\left[1+\frac{L_e}{l_d}\left(1-\frac{L_{MD}}{l_d}\right)\right] & -\frac{\omega L_e L_{MD}}{l_d l_F} & -\frac{\omega L_e L_{MD}}{l_d l_D} & -\frac{\hat{R}}{l_q}\left(1-\frac{L_{MQ}}{l_q}\right) & \frac{\hat{R}L_{MQ}}{l_q l_Q} & 0 & 0 \\
0 & 0 & 0 & \frac{r_Q L_{MQ}}{l_Q l_q} & -\frac{r_Q}{l_Q}\left(1-\frac{L_{MQ}}{l_Q}\right) & 0 & 0 \\
\hline
-\frac{L_{MD}}{3\tau_j l_d^2}\lambda_q & -\frac{L_{MD}}{3\tau_j l_d l_F}\lambda_q & \frac{L_{MD}}{3\tau_j l_d l_D}\lambda_q & \frac{L_{MQ}}{3\tau_j l_d^2}\lambda_q & \frac{L_{MQ}}{3\tau_j l_d l_Q}\lambda_q & -\frac{D}{\tau_j} & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0
\end{array}\right], \tag{68}
$$

and

$$
D = \begin{bmatrix} \sqrt{3}V_{infty}\sin(\delta - alpha) \\ v_F \\ 0 \\ -\sqrt{3}V_{infty}\cos(\delta-\alpha) \\ 0 \\ \frac{T_m}{\tau_j} \\ -1 \end{bmatrix} \tag{69}
$$

Appendix

Reponses of some 2^{nd}-order systems with simple state equations

With trigonometric functions in the state equations

With systems described by an ordinary differential equation

$$\ddot{x} \pm \chi x = \pm \kappa, \tag{70}$$

where χ is a trigonometric function and $\kappa < 0$ is a constant. Written in state equation form the system is

$$\begin{aligned} \dot{x}_1 &= x_2 & (71)\\ \dot{x}_2 &= \mp \chi\left(x_1\right) + \kappa. & (72) \end{aligned}$$

Figure 55 shows the result when $\chi\left(x_1\right) = \sin x_1$, $\kappa = 2.7$ for Figure 55 (a) and $\kappa = -2.7$ for Figure 55 (b). The result when $\ddot{x} - \sin x = \pm\kappa$ is similar to that when $\ddot{x} + \sin x = \pm\kappa$, and thus is not shown here. Likewise, the result when $\ddot{x} \pm \cos x = \pm\kappa$ is similar to that when $\ddot{x} \pm \sin x = \pm\kappa$, and therefore in not shown here either. The plots of the state variables start from the initial point of $(-5, 5)$.

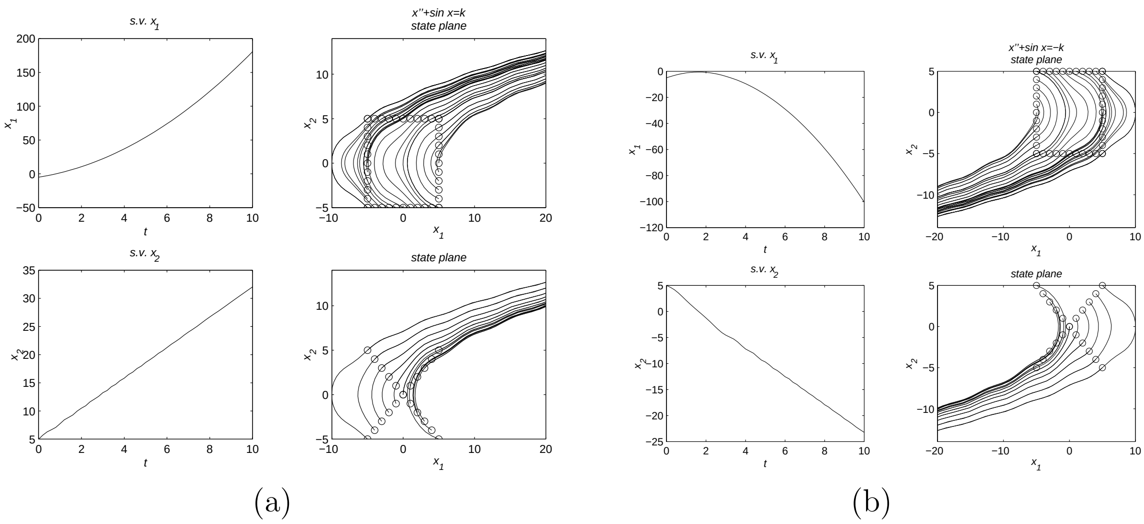
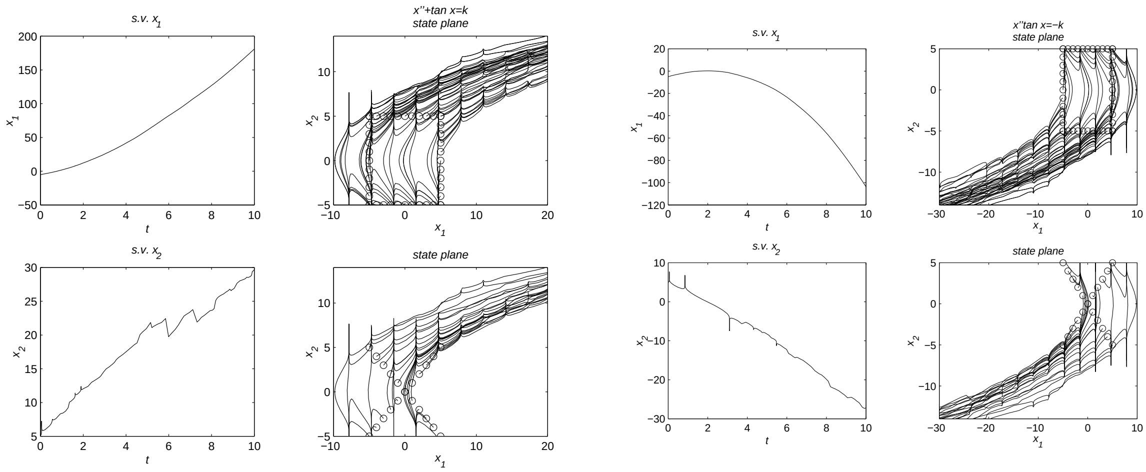


Figure 55: *State variables and state planes of (a) $\ddot{x} + \sin x = 2.7$, and (b) $\ddot{x} + \sin x = -2.7$*

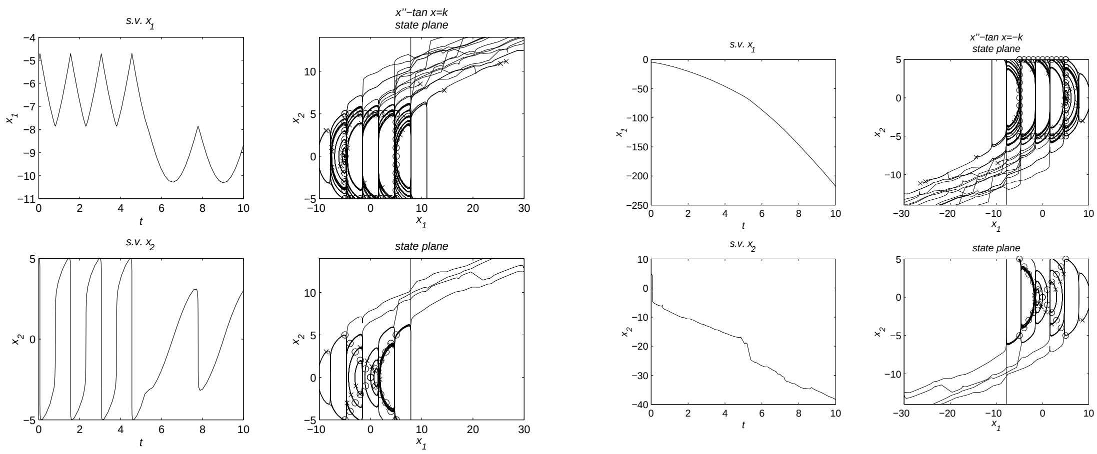
Figure 56 shows the result when $\chi(x_1) = \tan x_1$, $\kappa = 2.7$ for Figure 56 (a) and $\kappa = -2.7$ for Figure 56 (b). Figure 56 came across singularities while simulating.



(a) (b)

Figure 56: *State variables and state planes of (a)* $\ddot{x} + \tan x = 2.7$*, and (b)* $\ddot{x} + \tan x = -2.7$

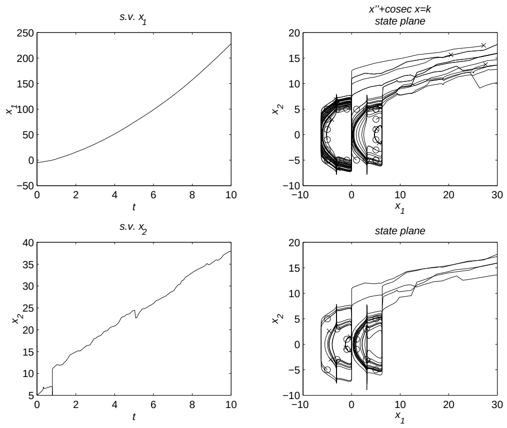
Figure 57 shows the result when $\chi(x_1) = -\tan x_1$, $\kappa = 2.7$ for Figure 57 (a) and $\kappa = -2.7$ for Figure 57 (b).



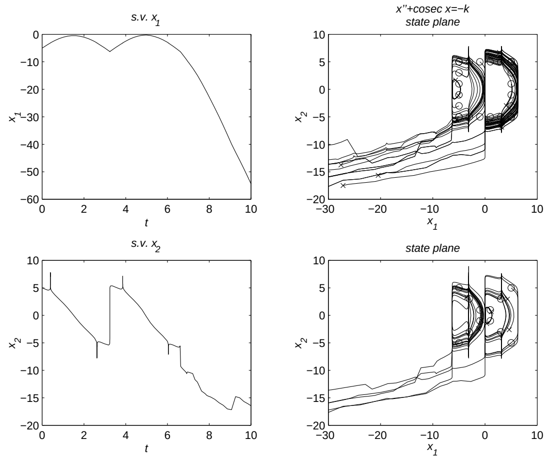
(a) (b)

Figure 57: *State variables and state planes of (a)* $\ddot{x} - \tan x = 2.7$*, and (b)* $\ddot{x} - \tan x = -2.7$

Figure 58 shows the result when $\chi(x_1) = \csc x_1$, $\kappa = 2.7$ for Figure 58 (a) and $\kappa = -2.7$ for Figure 58 (b).



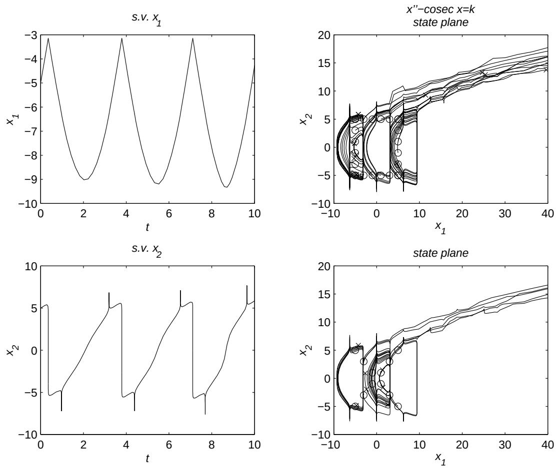
(a)



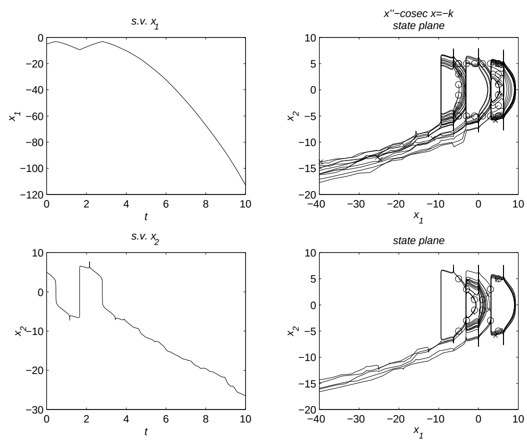
(b)

Figure 58: *State variables and state planes of (a)* $\ddot{x} + \csc x = 2.7$*, and (b)* $\ddot{x} + \csc x = -2.7$

Figure 59 shows the result when $\chi(x_1) = -\csc x_1$, $\kappa = 2.7$ for Figure 59 (a) and $\kappa = -2.7$ for Figure 59 (b).



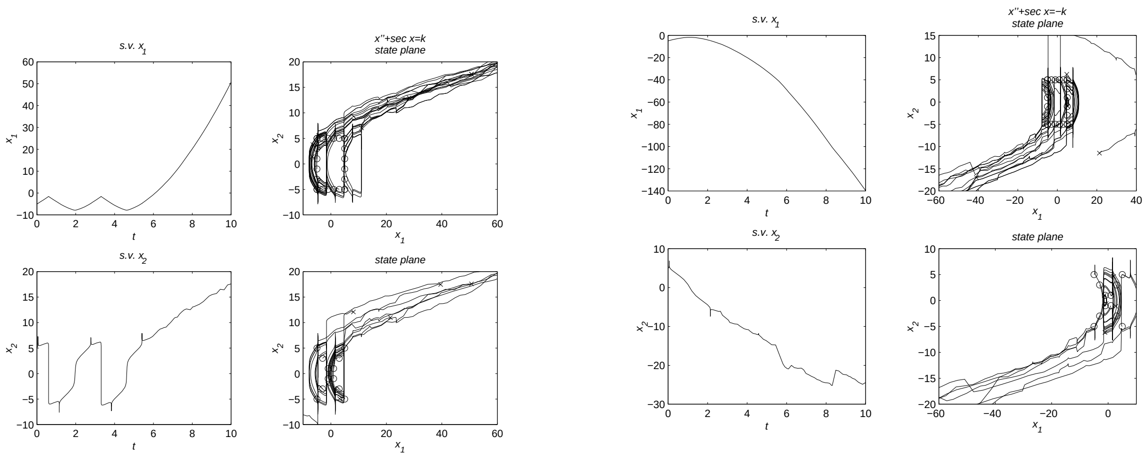
(a)



(b)

Figure 59: *State variables and state planes of (a)* $\ddot{x} - \csc x = 2.7$*, and (b)* $\ddot{x} - \csc x = -2.7$

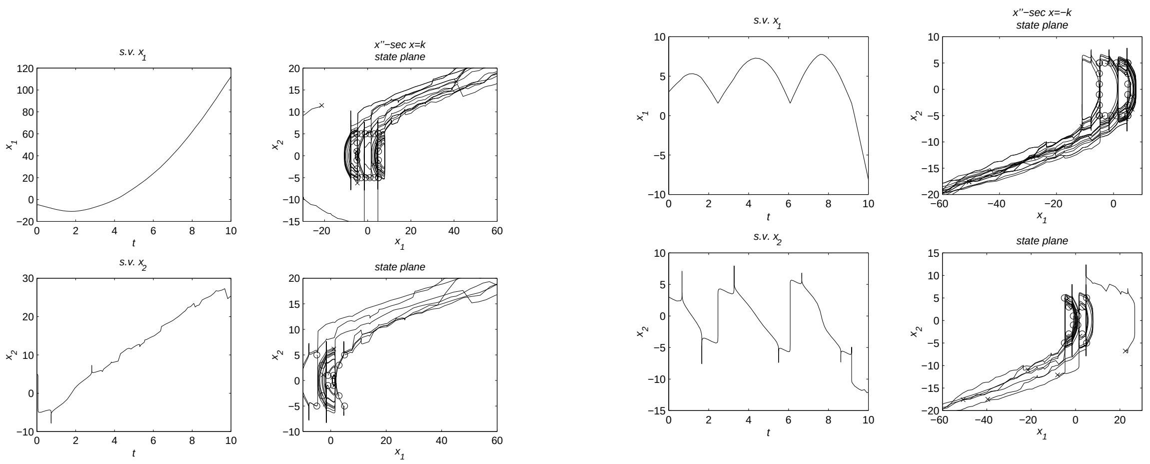
Figure 60 shows the result when $\chi(x_1) = \sec x_1$, $\kappa = 2.7$ for Figure 60 (a) and $\kappa = -2.7$ for Figure 60 (b).



(a) (b)

Figure 60: *State variables and state planes of (a)* $\ddot{x} + \sec x = 2.7$*, and (b)* $\ddot{x} + \sec x = -2.7$

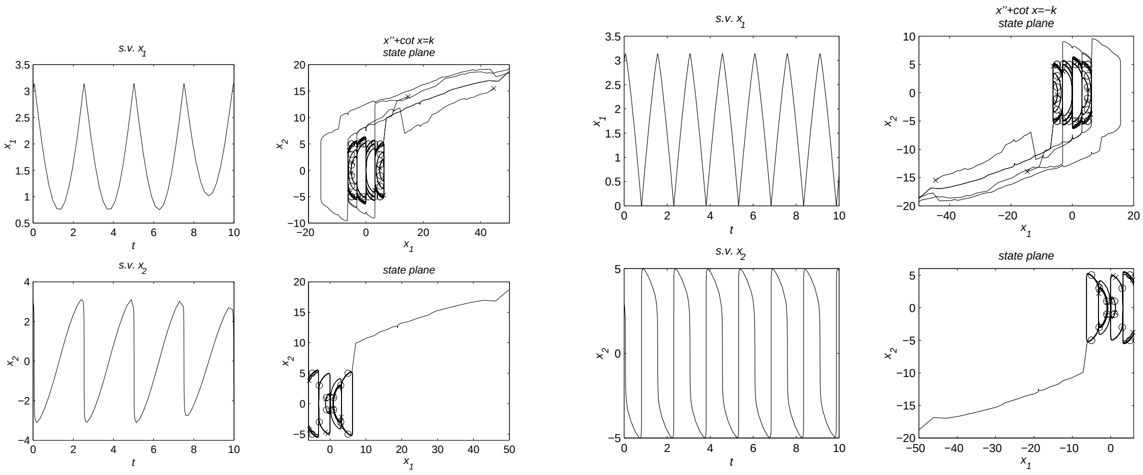
Figure 61 shows the result when $\chi(x_1) = -\sec x_1$, $\kappa = 2.7$ for Figure 61 (a) and $\kappa = -2.7$ for Figure 61 (b).



(a) (b)

Figure 61: *State variables and state planes of (a)* $\ddot{x} - \sec x = 2.7$*, and (b)* $\ddot{x} - \sec x = -2.7$ *The initial condition for the state variable plots in (b) is* $(3, 3)$

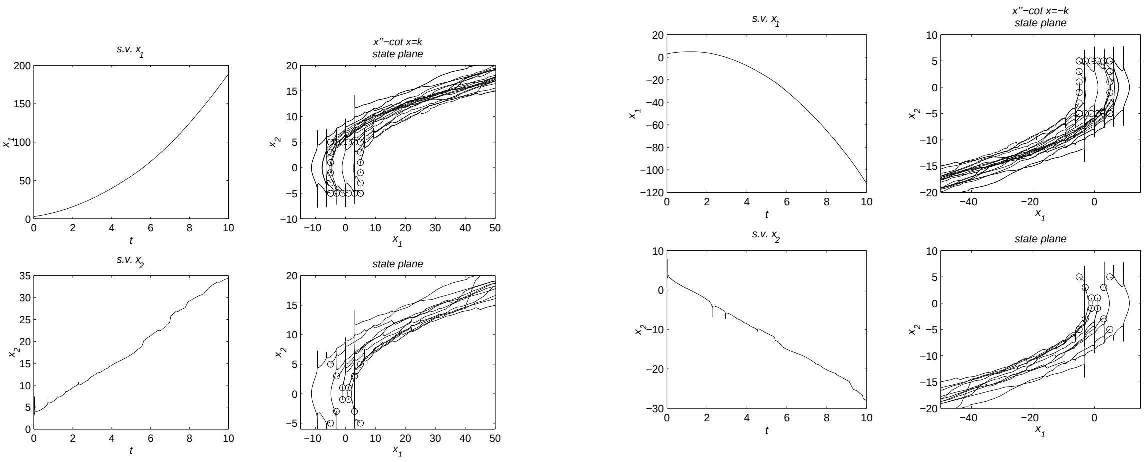
Figure 62 shows the result when $\chi\left(x_1\right) = \cot x_1$, $\kappa = 2.7$ for Figure 62 (a) and $\kappa = -2.7$ for Figure 62 (b).



(a) (b)

Figure 62: *State variables and state planes of (a)* $\ddot{x} + \cot x = 2.7$*, and (b)* $\ddot{x} + \cot x = -2.7$ *The initial condition for the state variable plots in is* $(3, 3)$

Figure 63 shows the result when $\chi\left(x_1\right) = -\cot x_1$, $\kappa = 2.7$ for Figure 63 (a) and $\kappa = -2.7$ for Figure 63 (b). (singularities were encountered while simulating Figure 63)



(a) (b)

Figure 63: *State variables and state planes of (a)* $\ddot{x} - \cot x = 2.7$*, and (b)* $\ddot{x} - \cot x = -2.7$ *The initial condition for the state variable plots in is* $(3, 3)$

Figure 64 shows the result when $\chi(x_1) = \sin^{-1} x_1$, $\kappa = 2.7$ for Figure 64 (a) and $\kappa = -2.7$ for Figure 64 (b).

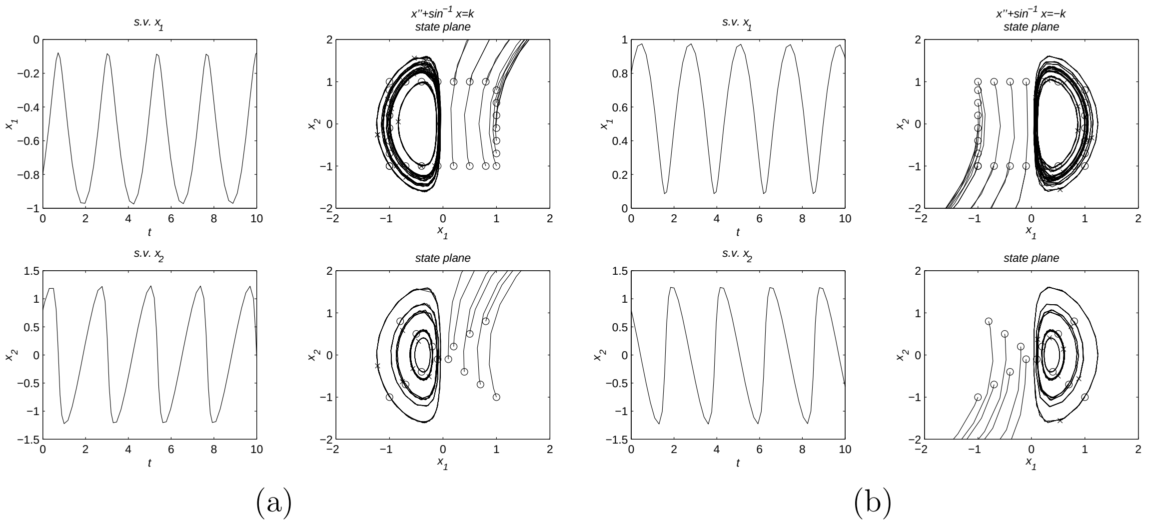


Figure 64: *State variables and state planes of (a)* $\ddot{x} + \sin^{-1} = 2.7$*, and (b)* $\ddot{x} + \sin^{-1} x = -2.7$ *The initial condition for the state variable plots in is* $(-0.8, 0.8)$ *for (a) and* $(0.8, 0.8)$ *for (b)*

Figure 65 shows the result when $\chi(x_1) = -\sin^{-1} x_1$, $\kappa = 2.7$ for Figure 65 (a) and $\kappa = -2.7$ for Figure 65 (b). (singularities were encountered while simulating Figure 65)

For $\chi(x_1) = \cos^{-1} x_1$ there are singularities within the equation and therefore could not simulate.

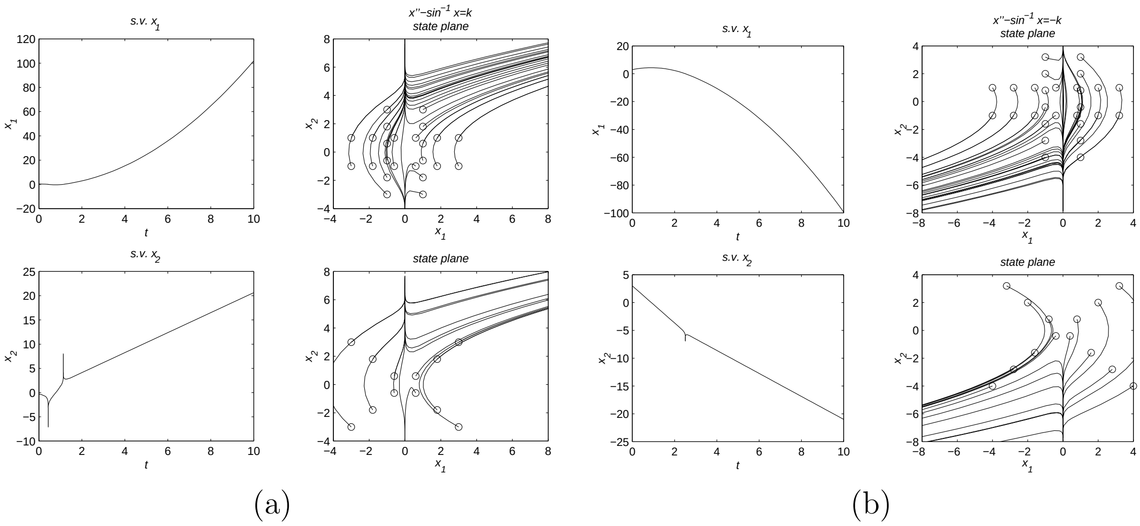
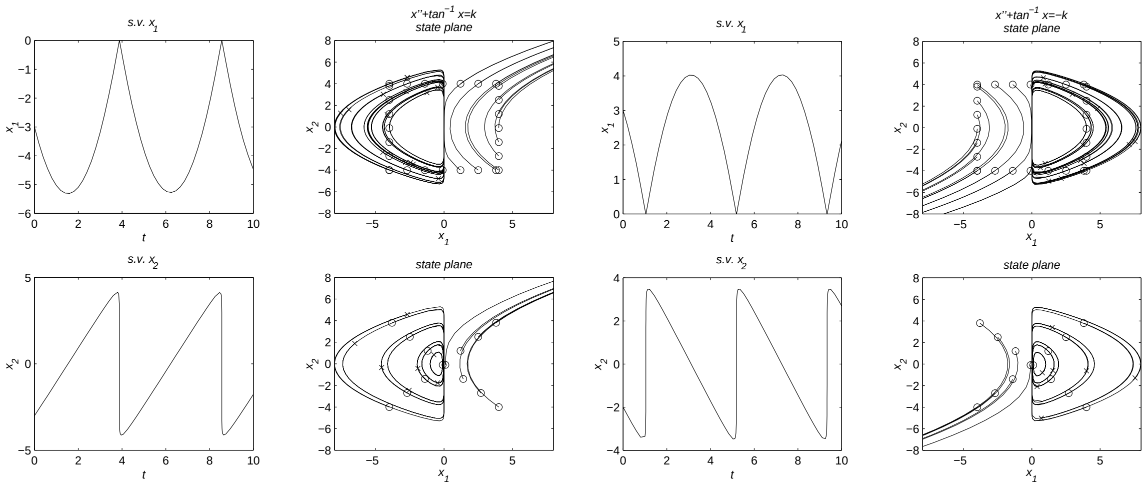


Figure 65: *State variables and state planes of (a)* $\ddot{x} - \sin^{-1} x = 2.7$ $t_{CPU} = 3.99\,\text{sec}$*, and (b)* $\ddot{x} - \sin^{-1} x = -2.7$ $t_{CPU} = 4.81\,\text{sec}$ *The initial condition for the state variable plots in is* $(-0.3, -0.3)$ *for (a)* $(3, 3)$ *for (b)*

Figure 66 shows the result when $\chi(x_1) = \tan^{-1} x_1$, $\kappa = 2.7$ for Figure 66 (a) and $\kappa = -2.7$ for Figure 66 (b).



(a) (b)

Figure 66: *State variables and state planes of (a)* $\ddot{x} + \tan^{-1} x = 2.7$ $t_{CPU} = 0.64\,\mathrm{sec}$*, and (b)* $\ddot{x} + \tan^{-1} x = -2.7$ $t_{CPU} = 0.61\,\mathrm{sec}$ *The initial condition for the state variable plots in is* $(-3, -3)$ *for (a) and* $(3, -2)$ *for (b)*

Figure 67 shows the result when $\chi(x_1) = -\tan^{-1} x_1$, $\kappa = 2.7$ for Figure 67 (a) and $\kappa = -2.7$ for Figure 67 (b). (integration tolerances were violated while simulating Figure 67)

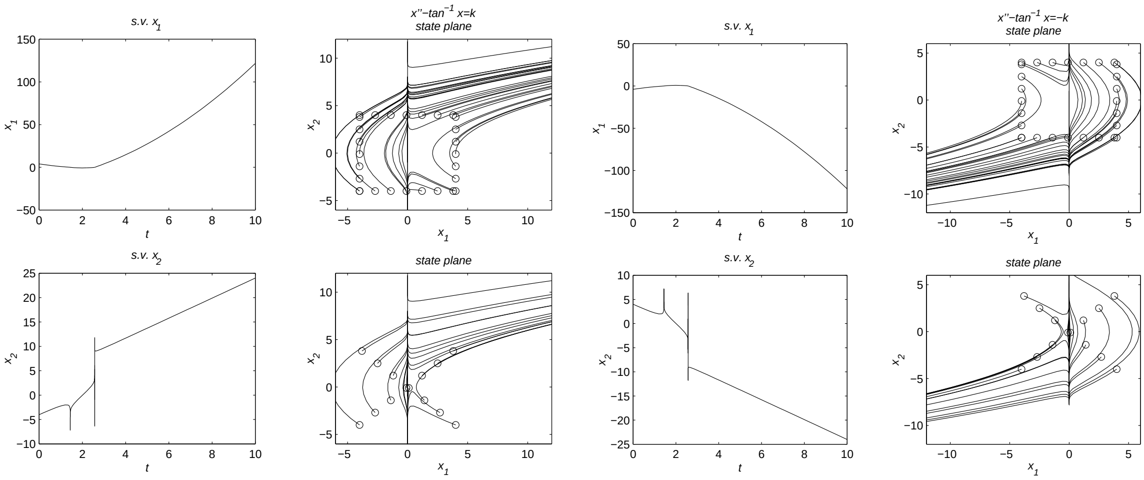
With polynomial of state variable terms in the state equations

Consider Equation 71 and 72 again but this time let

$$\chi(x) = x^n, \quad n = -5, -4, , -3, \ldots, 5.$$

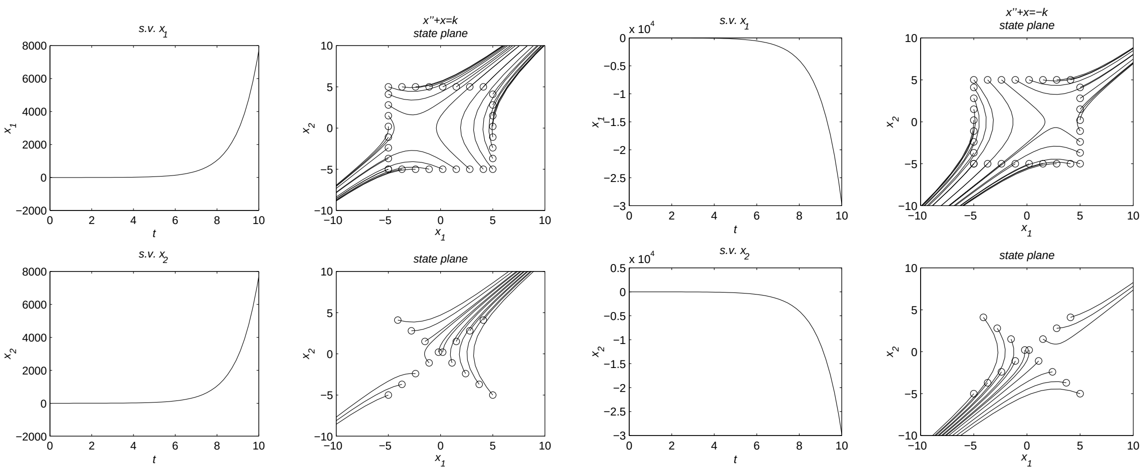
Figure 68 shows the result when $\chi(x_1) = x_1$, $\kappa = 2.7$ for Figure 68 (a) and $\kappa = -2.7$ for Figure 68 (b).

Figure 69 shows the result when $\chi(x_1) = -x_1$, $\kappa = 2.7$ for Figure 69 (a) and $\kappa = -2.7$ for Figure 69 (b). (integration tolerances were violated while simulating Figure 69)



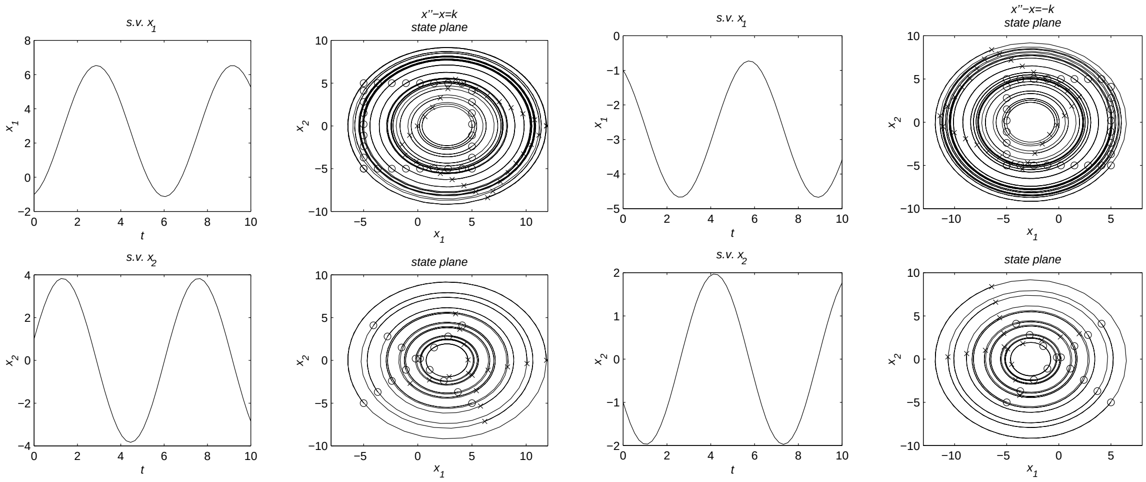
(a) (b)

Figure 67: *State variables and state planes of (a)* $\ddot{x} - \tan^{-1} x = 2.7$ $t_{CPU} = 0.75\,\mathrm{sec}$, *and (b)* $\ddot{x} - \tan^{-1} x = -2.7$ $t_{CPU} = 0.88\,\mathrm{sec}$ *The initial condition for the state variable plots in is* $(4, -4)$ *for (a)* $(-4, 4)$ *for (b)*



(a) (b)

Figure 68: *State variables and state planes of (a)* $\ddot{x} + x = 2.7$ $t_{CPU} = 0.41\,\mathrm{sec}$, *and (b)* $\ddot{x} + x = -2.7$ $t_{CPU} = 0.38\,\mathrm{sec}$ *The initial condition for the state variable plots in is* $(-1, -1)$ *for (a) and* $(-1, 1)$ *for (b)*



(a) (b)

Figure 69: *State variables and state planes of (a)* $\ddot{x} - x = 2.7$ $t_{CPU} = 0.41\,\mathrm{sec}$, *and (b)* $\ddot{x} - x = -2.7$ $t_{CPU} = 0.38\,\mathrm{sec}$ *The initial condition for the state variable plots in is* $(-1, 1)$ *for (a)* $(-1, -1)$ *for (b)*

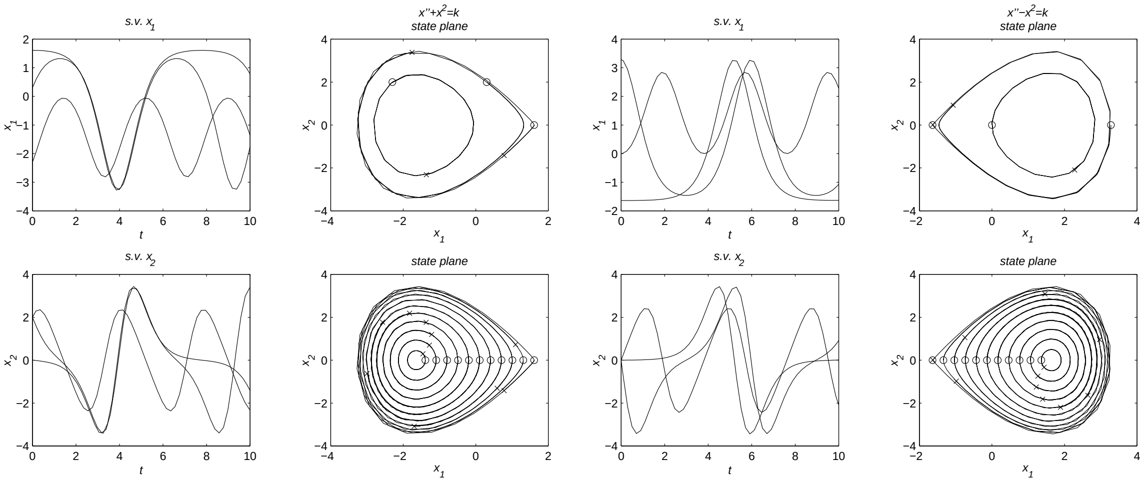
Figure 70 shows the result when $\chi(x_1) = x_1^2$, $\kappa = 2.7$ for Figure 70 (a) and $\kappa = -2.7$ for Figure 70 (b). When $\ddot{x} + x^2 = 2.7$ there are singularities within the solution and could not simulate. The same thing also happened with $\ddot{x} - x^2 = -2.7$, therefore neither of the two cases could be simulated. When $\ddot{x} + x^2 = -2.7$ simulation (Figure 70 (a)) showed that the response is stable approximately in the range of

$$-3.28 \leq x_1 \leq 1.61,$$

outside of which there appeared singularities in the solution. Likewise Figure 70 (b) and additional simulations showed the range of stability to be

$$-1.64 \leq x_1 \leq 3.28$$

for $\ddot{x} - x^2 = 2.7$.



(a) (b)

Figure 70: *State variables and state planes of (a) $\ddot{x}+x^2 = -2.7$ $t_{CPU} = 0.13\,\text{sec}$, and (b) $\ddot{x}-x^2 = 2.7$ $t_{CPU} = 0.14\,\text{sec}$*

When $\ddot{x} + x^3 = \pm 2.7$ there are singularities within the solution and could not be simulated. Figure 71 shows the result when $\ddot{x} - x^3 = \pm 2.7$

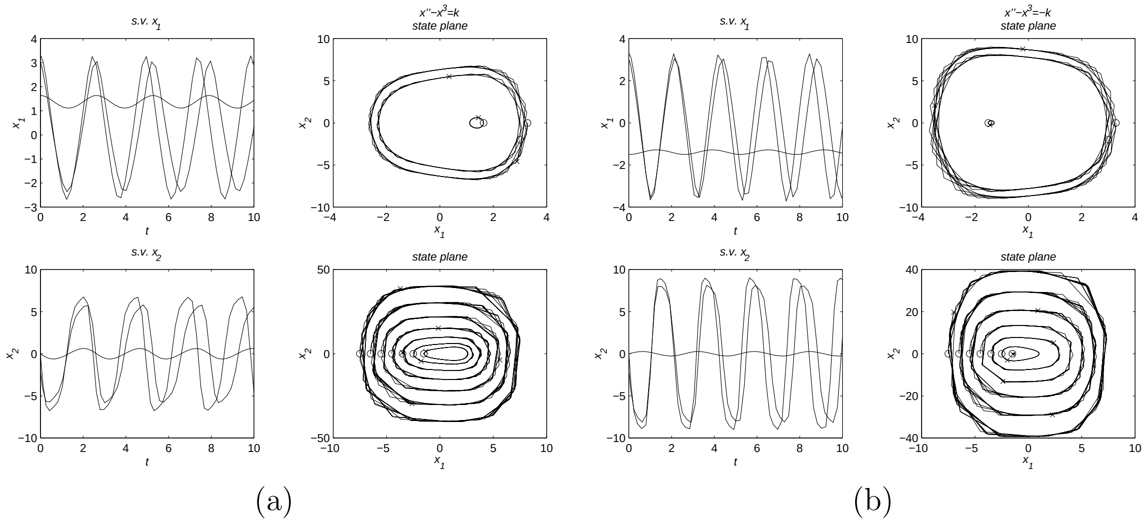


Figure 71: *State variables and state planes of (a)* $\ddot{x}-x^3 = 2.7$*, and (b)* $\ddot{x}-x^3 = -2.7$ $t_{CPU} = 0.15\,\mathrm{sec}$

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